

Lecture 10

Flight State Estimation

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What is it about?

QUESTION? states, output, performance signals imply that some states are not directly available. what shall we do about the hidden states?

ANSWER. guess by state estimation, its impact on control will also be addressed by integrated observer/control design.

Outline

- 1 Linear State Observer Design
- 2 Kalman Filter
- 3 Linear Quadratic Gaussian Design

1. Linear State Observer Design

- ▶ state estimation (observation) offers insight to the (internal) states
- ▶ based on output measurement: observability
- ▶ questions arise
 1. how accurate of the estimation
 2. how reliable of (estimated) state feedback in terms of stability, performance?
 3. robustness against noise

Observer of States

Consider the linear system:

$$\begin{cases} \dot{\underline{x}}(t) = \underline{A}\underline{x}(t) + \underline{B}\underline{u}(t) \\ \underline{y}(t) = \underline{C}\underline{x}(t) \end{cases}$$

The state observer is given by

$$\dot{\hat{\underline{x}}} = \underline{A}\hat{\underline{x}} + \underline{B}\underline{u} + \underline{L}(\underline{y} - \underline{C}\hat{\underline{x}}) = \underline{A}\hat{\underline{x}} + \underline{B}\underline{u} + \underline{L}(\underline{y} - \hat{\underline{y}})$$

re-organizing

$$\dot{\underline{\hat{x}}} = (\underline{A} - \underline{LC})\underline{\hat{x}} + \underline{Bu} + \underline{Ly}$$

Define mismatch error (not deviation)

$$\underline{\tilde{x}} = \underline{x} - \underline{\hat{x}}$$

$$\underline{\tilde{y}} = \underline{y} - \underline{\hat{y}}$$

leading to

$$\begin{aligned}\dot{\underline{\tilde{x}}} &= \dot{\underline{x}} - \dot{\underline{\hat{x}}} = \underline{Ax} + \underline{Bu} - (\underline{A} - \underline{LC})\underline{\hat{x}} - \underline{Bu} - \underline{Ly} \\ &= \underline{Ax} - (\underline{A} - \underline{LC})\underline{\hat{x}} - \underline{LCx} \\ &= (\underline{A} - \underline{LC})\underline{\tilde{x}}\end{aligned}$$

if $\mathbf{A}_o = (\mathbf{A} - \mathbf{LC})$ is asymptotically stable, for a given $\tilde{\mathbf{x}}_0$,

$$\tilde{\mathbf{x}}(t) \rightarrow 0, \quad \hat{\mathbf{x}}(t) \rightarrow \mathbf{x}(t), \quad \text{as } t \rightarrow \infty$$

Design of Observer Gain: L

recall LQR design (state feedback) $\underline{u} = -\mathbf{K}\underline{x}$, where

$$\dot{\underline{x}} = (\mathbf{A} - \mathbf{BK})\underline{x} = \mathbf{A}_c\underline{x}$$

$\mathbf{K}$ satisfies:

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}$$

$$\mathbf{0} = \mathbf{A}^T\mathbf{P} + \mathbf{PA} - \mathbf{PBR}^{-1}\mathbf{B}^T\mathbf{P}$$

and the closed-loop system is asymptotically stable when $\mathbf{A}$, $\mathbf{B}$ is controllable. now treat

$$(\mathbf{A} - \mathbf{LC})^T = \mathbf{A}^T - \mathbf{C}^T\mathbf{L}^T$$

$$\mathbf{A} \leftarrow \mathbf{A}^T$$

$$\mathbf{B} \leftarrow \mathbf{C}^T$$

$$\mathbf{K} \leftarrow \mathbf{L}^T$$

$$\begin{aligned} L^T &= R^{-1}CP \\ 0 &= A^T P + PA - PC^T R^{-1}CP \end{aligned}$$

the controllability becomes observability, defined by the Observability Matrix

$$Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

The system is observable iff $\text{rank}(Q) = n$.

2. Kalman Filter

Consider the linear system:

$$\begin{cases} \dot{\underline{x}}(t) = \mathbf{A}\underline{x}(t) + \mathbf{B}\underline{u}(t) + \mathbf{G}\underline{w}(t) \\ \underline{y}(t) = \mathbf{C}\underline{x}(t) + \underline{v}(t) \end{cases}$$

where $\underline{w}$ represents an unknown process noise, $\underline{v}$ represents the unknown measurement noise

$$\underline{w}(t) \sim (0, Q) - \text{mean, covariance}$$

$$\underline{v}(t) \sim (0, R) - \text{mean, covariance}$$

Random Process

For a random variable x , its *expected value (mean value)* is denoted by

$$\mu = \mathcal{E}\{x\} = \int_{-\infty}^{+\infty} \xi p(\xi) d\xi$$

where $p(\xi)$ is the *probability density function* expressing a given pattern of the random distribution property. Its *variance* value is defined by σ^2

$$\sigma^2 = \mathcal{E}\{(x - \mu)^2\}$$

Gaussian Distribution

A special random variable with a so-called *normal* (*Gaussian*) distribution property is that the probability density is completely determined by the mean value μ and variance σ^2 as follows:

$$p(\xi) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left[-\frac{(\xi - \mu)^2}{2\sigma^2} \right]$$

If we are dealing with two random variables x, y , where their mean values are μ_x, μ_y respectively, the *covariance* between these two variables is given by

$$\mathbf{cov}(x, y) = \mathcal{E}\{(x - \mu_x)(y - \mu_y)\}$$

When variables x, y are *uncorrelated*, i.e., $\mathcal{E}\{xy\} = \mathcal{E}\{x\}\mathcal{E}\{y\}$, its covariance value becomes zero: $\mathbf{cov}(x, y) = 0$.

We expand the random variable concepts to address a random vector. Given a random vector $\underline{x} \in \mathcal{R}^n$, we denote by $p(\underline{\xi})$ the probability density function over $\underline{x}$. It represents the probability that $\underline{x}$ takes on a value within the differential region $d\xi_1, \dots, d\xi_n$ centered at $\underline{\xi}$. Although the value of $\underline{x}$ is random, it is expected to know the mean value (or expected value) of $\underline{x}$ by

$$\underline{\mu} = \mathcal{E}\{\underline{x}\} = \int_{-\infty}^{+\infty} \underline{\xi} p(\underline{\xi}) d\underline{\xi}$$

The covariance of $\underline{x}$ is given by

$$\mathbf{\Sigma} = \mathcal{E}\{(\underline{x} - \underline{\mu})(\underline{x} - \underline{\mu})^T\} = [\Sigma_{ij}] = [\mathbf{cov}(x_i, x_j)]$$

Obviously, $\mathbf{\Sigma} \in \mathcal{R}^{n \times n}$. Again, a special random vector of normal (Gaussian) distribution property is that the probability density function is completely determined by the mean value $\underline{\mu}$ and the covariance matrix $\mathbf{\Sigma}$:

$$p(\underline{\xi}) = \frac{1}{(\sqrt{2\pi})^n |\mathbf{\Sigma}|^{1/2}} \exp \left[-\frac{1}{2} (\underline{\xi} - \underline{\mu})^T \mathbf{\Sigma}^{-1} (\underline{\xi} - \underline{\mu}) \right]$$

where $|\mathbf{\Sigma}|$ is the determinant of the matrix $\mathbf{\Sigma}$.

Another special process that is relevant to our discussion is called the *white noise process*, when the process has the property of zero mean value and uncorrelated feature, that is:

$$\mathcal{E}\{x(t_i)\} = 0$$
$$\mathbf{cov}(x(t_i), x(t_j)) = \begin{cases} Q \geq 0 & t_i = t_j \\ 0 & t_i \neq t_j \end{cases}$$

Consider the linear system:

$$\begin{cases} \dot{\underline{x}}(t) = \underline{A}\underline{x}(t) + \underline{B}\underline{u}(t) + \underline{G}\underline{w}(t) \\ \underline{y}(t) = \underline{C}\underline{x}(t) + \underline{v}(t) \end{cases}$$

where $\underline{w}$ represents an unknown process noise, $\underline{v}$ represents the unknown measurement noise

$$\underline{w}(t) \sim (0, Q) - \text{mean, covariance}$$

$$\underline{v}(t) \sim (0, R) - \text{mean, covariance}$$

$$\begin{aligned}
 \dot{\tilde{\underline{x}}} &= \dot{\underline{x}} - \dot{\hat{\underline{x}}} \\
 &= (\mathbf{A} - \mathbf{LC})\tilde{\underline{x}} + \mathbf{G}\underline{w} - \mathbf{L}\underline{v} \\
 \tilde{\underline{y}} &= \mathbf{C}\tilde{\underline{x}} + \underline{v}
 \end{aligned}$$

leading to

$$\mathcal{E}\{\dot{\tilde{\underline{x}}}\} = \mathbf{A}_o \mathcal{E}\{\tilde{\underline{x}}\} + \mathbf{G}\mathcal{E}\{\underline{w}\} - \mathbf{L}\mathcal{E}\{\underline{v}\} = \mathbf{A}_o \mathcal{E}\{\tilde{\underline{x}}\}$$

$$\mathcal{E}\{\tilde{\underline{x}}\} \rightarrow 0$$

when $\mathbf{A}_o$ stable, that is

$$\mathcal{E}\{\underline{x}\} \rightarrow \hat{\underline{x}}$$

treat the error covariance

$$\mathbf{P} = \mathcal{E}\{\underline{\tilde{\mathbf{x}}}\underline{\tilde{\mathbf{x}}}^T\}$$

some mathematical results (without proof)

$$\mathcal{E}\{\underline{\mathbf{v}}\underline{\tilde{\mathbf{x}}}^T\} = -\frac{1}{2}\mathbf{R}\mathbf{L}^T$$

$$\mathcal{E}\{\underline{\mathbf{w}}\underline{\tilde{\mathbf{x}}}^T\} = \frac{1}{2}\mathbf{Q}\mathbf{G}^T$$

$$\dot{\mathbf{P}} = \mathcal{E}\{\dot{\underline{\tilde{\mathbf{x}}}}\underline{\tilde{\mathbf{x}}}^T\} + \mathcal{E}\{\underline{\tilde{\mathbf{x}}}\dot{\underline{\tilde{\mathbf{x}}}}^T\}$$

$$= \mathcal{E}\{\mathbf{A}_o\underline{\tilde{\mathbf{x}}}\underline{\tilde{\mathbf{x}}}^T + \mathbf{G}\underline{\mathbf{w}}\underline{\tilde{\mathbf{x}}}^T - \mathbf{L}\underline{\mathbf{v}}\underline{\tilde{\mathbf{x}}}^T\} + \mathcal{E}\{\underline{\mathbf{x}}\dot{\underline{\tilde{\mathbf{x}}}}^T\}$$

$$= \mathbf{A}_o\mathbf{P} + \frac{1}{2}\mathbf{G}\mathbf{Q}\mathbf{G}^T + \frac{1}{2}\mathbf{L}\mathbf{R}\mathbf{L}^T + \mathbf{P}\mathbf{A}_o + \frac{1}{2}\mathbf{G}\mathbf{Q}\mathbf{G}^T + \frac{1}{2}\mathbf{L}\mathbf{R}\mathbf{L}^T$$

$$= \mathbf{A}_o\mathbf{P} + \mathbf{P}\mathbf{A}_o^T + \mathbf{L}\mathbf{R}\mathbf{L}^T + \mathbf{G}\mathbf{Q}\mathbf{G}^T$$

This equation looks similar to the closed-loop Riccati equation in linear quadratic control design. From design perspective, it is equivalent to solve the following algebraic Riccati equation if $\mathbf{P}$ is constant:

$$\mathbf{AP} + \mathbf{PA}^T + \mathbf{GQG}^T - \mathbf{PC}^T \mathbf{R}^{-1} \mathbf{CP} = 0$$

and the observer matrix becomes the Kalman gain

$$\mathbf{L} = \mathbf{PC}^T \mathbf{R}^{-1}$$

If the noise is modeled:

$$\begin{cases} \dot{\underline{x}}_w(t) = \mathbf{A}_w \underline{x}_w(t) + \mathbf{B}_w \underline{n}(t) \\ \underline{w}(t) = \mathbf{C}_w \underline{x}_w(t) + \mathbf{D}_w \underline{n}(t) \end{cases}$$

then the augmented system becomes:

$$\begin{bmatrix} \dot{\underline{x}} \\ \dot{\underline{x}}_w \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{G}\mathbf{C}_w \\ & \mathbf{A}_w \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{x}_w \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \underline{u} + \begin{bmatrix} \mathbf{G}\mathbf{D}_w \\ \mathbf{B}_w \end{bmatrix} \underline{n}$$

$$\underline{y} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{x}_w \end{bmatrix} + \underline{v}$$

[Example 1] Consider the short-period flight dynamics for pitch-rate (tracking) control, and limit the control for δ_e only ([stevens2003], Section 6.4 with modification)

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} -1.01887 & 0.90506 \\ 0.82225 & -1.07741 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} -0.00215 \\ -0.17555 \end{bmatrix} \delta_e + \begin{bmatrix} 0.00203 \\ -0.00164 \end{bmatrix} w_g$$

there is a disturbance of vertical wind gust w_g applied to the flight. The gust noise is not white, rather defined by a so-called Dryden model, that a spectral density is given as the gust

$$\phi_w(\omega) = 2L\sigma^2 \frac{1 + 3L^2\omega^2}{(1 + L^2\omega^2)^2}$$

of turbulence intensity σ , L the turbulence scale length divided by true airspeed

$$H_w(s) = \sigma \sqrt{\frac{6}{L}} \frac{s + \frac{1}{L\sqrt{3}}}{s^2 + \frac{2}{L}s + \frac{1}{L^2}}$$

if we choose $L = 3.39$, $\sigma = 10$,

$$\begin{aligned} \begin{bmatrix} \dot{x}_{w,1} \\ \dot{x}_{w,2} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -\frac{1}{L^2} & -\frac{2}{L} \end{bmatrix} \begin{bmatrix} x_{w,1} \\ x_{w,2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w \\ w_g &= \sigma \sqrt{\frac{6}{L}} \begin{bmatrix} \frac{1}{L\sqrt{3}} & 1 \end{bmatrix} \begin{bmatrix} x_{w,1} \\ x_{w,2} \end{bmatrix} \end{aligned}$$

the augmented system:

$$\begin{bmatrix} \dot{\underline{x}} \\ \dot{\underline{x}}_w \\ \dot{\delta}_e \end{bmatrix} = \left[\begin{array}{c|c|c} \mathbf{A} & \mathbf{GC}_w & \mathbf{B} \\ \hline \mathbf{0} & \mathbf{A}_w & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & -20.2 \end{array} \right] \begin{bmatrix} \underline{x} \\ \underline{x}_w \\ \delta_e \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ 20.2 \end{bmatrix} u + \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_w \\ \mathbf{0} \end{bmatrix} w$$

$$\underline{y} = \begin{bmatrix} n_z \\ q \end{bmatrix} = \begin{bmatrix} 15.87875 & 1.48113 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ x_{w,1} \\ x_{w,2} \\ \delta_e \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

with

$$\underline{w}(t) \sim (0, \mathbf{Q}) - \mathbf{Q} = 1$$

$$\underline{v}(t) \sim (0, \mathbf{R}) - \mathbf{R} = \begin{bmatrix} 1/20 & \\ & 1/60 \end{bmatrix}$$

linear quadratic design

$$\begin{aligned}0 &= AP + PA^T + GQG^T - PC^TR^{-1}CP \\ L &= PC^TR^{-1}\end{aligned}$$

leading to

$$L = \begin{bmatrix} 0.0375 & -0.0041 \\ -0.0202 & 0.0029 \\ 3.6010 & -0.2427 \\ 1.9074 & -0.2872 \\ 0 & 0 \end{bmatrix}$$

3. Linear Quadratic Gaussian Design

- ▶ integrated state feedback and state observer design

Consider the linear system with noise again

$$\begin{cases} \dot{\underline{x}}(t) = \mathbf{A}\underline{x}(t) + \mathbf{B}\underline{u}(t) + \mathbf{G}\underline{w}(t) \\ \underline{y}(t) = \mathbf{C}\underline{x}(t) + \underline{v}(t) \end{cases}$$

apply the state observer:

$$\dot{\underline{\hat{x}}} = \mathbf{A}\underline{\hat{x}} + \mathbf{B}\underline{u} + \mathbf{L}(\underline{y} - \mathbf{C}\underline{\hat{x}}) = (\mathbf{A} - \mathbf{L}\mathbf{C})\underline{\hat{x}} + \mathbf{B}\underline{u} + \mathbf{L}\underline{y}$$

and the control law:

$$\underline{u} = -\mathbf{K}\underline{\hat{x}}$$

define

$$\underline{\tilde{x}} = \underline{x} - \underline{\hat{x}}$$

$$\underline{u} = -\underline{K}\underline{\hat{x}} = -\underline{K}(\underline{x} - \underline{\tilde{x}})$$

then we have

$$\begin{aligned}\dot{\underline{x}} &= \underline{A}\underline{x} + \underline{B}\underline{u} + \underline{G}\underline{w} \\ &= (\underline{A} - \underline{BK})\underline{x} + \underline{BK}\underline{\tilde{x}} + \underline{G}\underline{w} \\ \dot{\underline{\tilde{x}}} &= \dot{\underline{x}} - \dot{\underline{\hat{x}}} \\ &= (\underline{A} - \underline{BK})\underline{x} + \underline{BK}\underline{\tilde{x}} + \underline{G}\underline{w} - \\ &\quad \left[(\underline{A} - \underline{LC})\underline{\hat{x}} - \underline{BK}(\underline{x} - \underline{\tilde{x}}) + \underline{LC}\underline{x} + \underline{L}\underline{v} \right] \\ &= (\underline{A} - \underline{LC})\underline{\tilde{x}} + \underline{G}\underline{w} - \underline{L}\underline{v}\end{aligned}$$

the closed-loop system becomes:

$$\begin{bmatrix} \dot{\underline{x}} \\ \dot{\underline{\tilde{x}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{BK} \\ \mathbf{A} - \mathbf{LC} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{\tilde{x}} \end{bmatrix} + \begin{bmatrix} \mathbf{G} \\ \mathbf{G} \end{bmatrix} \underline{w} - \begin{bmatrix} \mathbf{0} \\ \mathbf{L} \end{bmatrix} \underline{v}$$

$$\underline{y} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{\tilde{x}} \end{bmatrix} + \underline{v}$$

LQG: The Separation Principle

based on the closed-loop system structure,

- ▶ to design $\mathbf{K}$ (LQR) for stabilization and tracking
- ▶ to design $\mathbf{L}$ (Kalman filter) for convergence of observation

[Example 2] Consider the short-period flight dynamics for pitch-rate (tracking) control, and limit the control for δ_e only ([stevens2003], Section 6.4 with modification)

$$\begin{aligned}\dot{\underline{x}}_s &= \begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = [\mathbf{A}_s] \begin{bmatrix} \alpha \\ q \end{bmatrix} + [\mathbf{B}_s]\delta_e + [\mathbf{G}_s]w_g \\ \underline{y} &= \begin{bmatrix} n_z \\ q \end{bmatrix} = \begin{bmatrix} 15.87875 & 1.48113 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= [\mathbf{C}_s]\underline{x}_s + \underline{v} \\ \dot{\underline{x}}_w &= [\mathbf{A}_w]\underline{x}_w + \mathbf{B}_w w \\ w_g &= [\mathbf{C}_w]\underline{x}_w \\ z = q &= \mathbf{H}\underline{x}_s\end{aligned}$$

Case 1: full-state feedback for tracking

$$\begin{aligned} \begin{bmatrix} \dot{\underline{x}}_s \\ \dot{\delta}_e \end{bmatrix} &= \begin{bmatrix} \mathbf{A}_s & \mathbf{B}_s \\ \mathbf{0} & -20.2 \end{bmatrix} \begin{bmatrix} \underline{x}_s \\ \delta_e \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ 20.2 \end{bmatrix} u_e \\ \dot{\underline{x}}_1 &= \mathbf{A}_1 \underline{x}_1 + \mathbf{B}_1 u_e \\ \mathbf{y} &= \begin{bmatrix} \mathbf{C}_s & 0 \end{bmatrix} \begin{bmatrix} \underline{x}_s \\ \delta_e \end{bmatrix} = \mathbf{C}_1 \underline{x}_1 \\ u_e &= -\mathbf{K}_1 \begin{bmatrix} \underline{x}_s \\ \delta_e \end{bmatrix} + r \end{aligned}$$

closed-loop step response of q

[missing figure]
figures/stevens6_4_2ii_fig1

Case 2: Tracking Control by output feedback

introduce

$$\dot{e} = e = r - z$$

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\delta}_e \\ \dot{e} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \\ \hline 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \delta_e \\ e \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \hline 0 \end{bmatrix} u_e + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$\dot{\underline{x}}_2 = \mathbf{A}_2 \underline{x}_2 + \mathbf{B}_2 u_e + \mathbf{E} r$$

$$\underline{y}_2 = \begin{bmatrix} n_z \\ q \\ e \end{bmatrix} = \begin{bmatrix} \mathbf{C}_1 & \begin{array}{c} 0 \\ 0 \end{array} \\ \hline \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \delta_e \\ e \end{bmatrix} = \mathbf{C}_2 \underline{x}_2$$

$$z = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \underline{x}_2 = \mathbf{H}_2 \underline{x}_2$$

$$u_e = -\mathbf{K}_2 \underline{y}_2$$

closed-loop step response of q

[missing figure]
figures/stevens6_4_2ii_fig2

Case 3: tracking control of n_z

skip

Case 4: Tracking Control by state feedback

introduce

$$\dot{e} = e = r - z$$

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\delta}_e \\ \dot{e} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & \begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix} \\ \hline \begin{smallmatrix} 0 & -1 & 0 \end{smallmatrix} & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \delta_e \\ e \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \hline 0 \end{bmatrix} u_e + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$\dot{\underline{x}}_2 = \mathbf{A}_2 \underline{x}_2 + \mathbf{B}_2 u_e + \mathbf{E} r$$

$$\underline{y}_2 = \begin{bmatrix} n_z \\ q \\ e \end{bmatrix} = \begin{bmatrix} \mathbf{C}_1 & \begin{smallmatrix} 0 \end{smallmatrix} \\ \hline \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \delta_e \\ e \end{bmatrix} = \mathbf{C}_2 \underline{x}_2$$

$$z = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \underline{x}_2 = \mathbf{H}_2 \underline{x}_2$$

$$u_e = -\mathbf{K}_4 \underline{x}_2$$

closed-loop step response of q

[missing figure]
figures/stevens6_4_2ii_fig4

Case 5: Kalman Filter

$$\dot{\underline{x}}_2 = \mathbf{A}_2 \underline{x}_2 + \mathbf{B}_2 u_e + \mathbf{E} r + \mathbf{G} w_g$$

$$\underline{y}_2 = \mathbf{C}_2 \underline{x}_2 + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\begin{bmatrix} \dot{f}_1 \\ \dot{f}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{L^2} & -\frac{2}{L} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w$$

$$w_g = \sigma \sqrt{\frac{6}{L}} \begin{bmatrix} \frac{1}{L\sqrt{3}} & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, L = 3.49, \sigma = 10$$

$$\mathbf{Q} = 1$$

$$\mathbf{R} = \begin{bmatrix} 1/20 & & \\ & 1/60 & \\ & & 1 \end{bmatrix}$$

Augmented system:

$$\begin{bmatrix} \dot{\underline{x}}_2 \\ \underline{\dot{f}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_2 & \mathbf{G}\mathbf{C}_w \\ & \mathbf{A}_w \end{bmatrix} \begin{bmatrix} \underline{x}_2 \\ \underline{f} \end{bmatrix} + \begin{bmatrix} \mathbf{B}_2 \\ \mathbf{0} \end{bmatrix} u_e + \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_w \end{bmatrix} w + \begin{bmatrix} \mathbf{E} \\ \mathbf{0} \end{bmatrix} r$$

$$\dot{\underline{x}}_5 = \mathbf{A}_5 \underline{x}_5 + \mathbf{B}_5 u_e + \mathbf{E}_5 r + \mathbf{G}_5 w$$

$$\underline{y}_2 = \begin{bmatrix} \mathbf{C}_2 & \mathbf{0} \end{bmatrix} \begin{bmatrix} \underline{x}_2 \\ \underline{f} \end{bmatrix} + \underline{v}$$

$$\underline{y}_5 = \mathbf{C}_5 \underline{x}_5 + \underline{v}$$

$$\dot{\hat{\underline{x}}}_5 = \mathbf{A}_5 \hat{\underline{x}}_5 + \mathbf{B}_5 u_e + \mathbf{L}_5 (\underline{y}_5 - \mathbf{C}_5 \hat{\underline{x}}_5)$$

Case 6: LQG Design

$$\dot{\underline{x}}_5 = \underline{A}_5 \underline{x}_5 + \underline{B}_5 u_e + \underline{E}_5 r + \underline{G}_5 w$$

$$\underline{y}_5 = \underline{C}_5 \underline{x}_5 + \underline{v}$$

$$\dot{\hat{\underline{x}}}_5 = \underline{A}_5 \hat{\underline{x}}_5 + \underline{B}_5 u_e + \underline{L}_5 (\underline{y}_5 - \underline{C}_5 \hat{\underline{x}}_5)$$

$$u_e = -\underline{K}_5 \hat{\underline{x}}_5$$

closed-loop step response of q , w/o noise

[missing figure]
figures/stevens6_4_2ii_fig5

Summary

- 1 Linear State Observer Design
- 2 Kalman Filter
- 3 Linear Quadratic Gaussian Design

Summary: What It is About

- ▶ LQG: integrated estimation and control
 - ▶ separation principle: design $\mathbf{K}$ and $\mathbf{L}$ respectively
 - ▶ Kalman filter: companion format of LQR

Further Readings

- ▶ Kalman filter and LQG [**stevens2003**] (ch6.4)

References

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