

Lecture 08

Robustness in Flight Control

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What is it about?

QUESTION? Modern control based flight control seems to be wonderful, however, it depends on the accuracy of model to represent the flight dynamics. What if model uncertainty, variations?

ANSWER. Robust Flight control addresses the robustness against these factors.

Outline

- 1 Robustness
- 2 Multivariable Frequency-domain Analysis
- 3 Robustness Performance Specifications
- 4 Robust Output Feedback Design

1. Robustness

Robustness is the capability against the presence of uncertainties and disturbances.

- ▶ (external) disturbances and (sensor) noises
 - ▶ environmental (external) disturbances, e.g. wind gust
 - ▶ internal disturbances, e.g. sensor measurement noise
- ▶ modelling errors or uncertainties
 - ▶ unmodelled high-frequency dynamics, e.g. flexible aircraft modes
 - ▶ parameter variation: low-frequency effect, e.g. linearization around equilibrium
- ▶ stability robustness is the ability of providing stability in spite of modeling errors or plant parameter variations
- ▶ performance robustness is the ability of guaranteeing acceptable performance in the presence of disturbances

2. Multivariable Frequency-domain Analysis

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- ▶ d - disturbance signal
- ▶ z - performance signal
- ▶ n - measurement noise
- ▶ r - command (reference) signal
- ▶ s - controller input, $s = r - z - n$
- ▶ u - control signal, $u = \mathbf{K}s$
- ▶ y - system output
- ▶ e - tracking error: $e = r - z$

closed-loop system

$$\begin{aligned}z &= \mathbf{GK}s + d \\ &= \mathbf{GK}(r - z - n) + d\end{aligned}$$

$$z = (\mathbf{1} + \mathbf{GK})^{-1}\mathbf{GK}r - (\mathbf{1} + \mathbf{GK})^{-1}\mathbf{GK}n + (\mathbf{1} + \mathbf{GK})^{-1}d$$

$$\begin{aligned}e = r - z &= [\mathbf{1} - (\mathbf{1} + \mathbf{GK})^{-1}\mathbf{GK}]r \\ &\quad + (\mathbf{1} + \mathbf{GK})^{-1}\mathbf{GK}n - (\mathbf{1} + \mathbf{GK})^{-1}d \\ &= (\mathbf{1} + \mathbf{GK})^{-1}r + (\mathbf{1} + \mathbf{GK})^{-1}\mathbf{GK}n - (\mathbf{1} + \mathbf{GK})^{-1}d \\ &= (\mathbf{1} + \mathbf{GK})^{-1}(r - d) + (\mathbf{1} + \mathbf{GK})^{-1}\mathbf{GK}n \\ &\quad (\mathbf{1} + \mathbf{GK})^{-1}\mathbf{GK} = \mathbf{GK}(\mathbf{1} + \mathbf{GK})^{-1}\end{aligned}$$

therefore,

$$z = \mathbf{GK}(\mathbf{1} + \mathbf{GK})^{-1}(r - n) + (\mathbf{1} + \mathbf{GK})^{-1}d$$

$$e = r - z = (\mathbf{1} + \mathbf{GK})^{-1}(r - d) + \mathbf{GK}(\mathbf{1} + \mathbf{GK})^{-1}n$$

Sensitivity

Define the system sensitivity and co-sensitivity

$$\mathbf{S} = (\mathbf{1} + \mathbf{GK})^{-1}$$

$$\mathbf{T} = \mathbf{GK}(\mathbf{1} + \mathbf{GK})^{-1} = (\mathbf{1} + \mathbf{GK})^{-1}\mathbf{GK}$$

$$\mathbf{S} + \mathbf{T} = \mathbf{1}$$

Loop gain

$$\mathbf{L} = \mathbf{GK}$$

and return difference

$$\mathbf{1} + \mathbf{L} = \mathbf{1} + \mathbf{GK}$$

again,

$$z = \mathbf{T}(r - n) + \mathbf{S}d$$

$$e = \mathbf{S}(r - d) + \mathbf{T}n$$

- ▶ at (low) frequency where d is large, (and n small), make $\mathbf{S}$ small and $\mathbf{T} \rightarrow \mathbf{1}$

$$z \rightarrow (r - n) \approx r, e \rightarrow n \approx 0$$

disturbance rejection

- ▶ at (high) frequency where n is large (and d small), make $\mathbf{T}$ small and $\mathbf{S} \rightarrow \mathbf{1}$

$$z \rightarrow d \approx 0, e \rightarrow r - d \approx r$$

noise rejection

in other words,

- ▶ S small at low frequency
- ▶ T small at high frequency

Singular Value

Given a real matrix $\mathbf{M} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$ where $\mathbf{U}, \mathbf{V}$ are orthogonal

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}$$

where $\sigma_1 \geq \dots \geq \sigma_r$ are singular values, r is the rank of $\mathbf{M}$.

$$\mathbf{M}\mathbf{M}^T = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T\mathbf{V}\mathbf{\Sigma}^T\mathbf{U}^T = \mathbf{U}\mathbf{\Sigma}\mathbf{\Sigma}^T\mathbf{U}^T$$

For a transfer matrix $\mathbf{G}(s)$, its singular values indicate the magnitude of $\mathbf{G}(j\omega)$

[Example 1] Consider the lateral flight dynamics [stevens2003] (ch6.2),

$$\begin{bmatrix} \dot{\beta} \\ \dot{\phi} \\ \dot{p} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} -0.3220 & 0.0640 & 0.0364 & -0.9917 \\ 0 & 0 & 1 & 0.0037 \\ -30.6492 & 0 & -3.6784 & 0.6646 \\ 8.5396 & 0 & -0.0254 & -0.4764 \end{bmatrix} \begin{bmatrix} \beta \\ \phi \\ p \\ r \end{bmatrix} + \begin{bmatrix} -0.9917 & 0.0003 \\ 0 & 0 \\ -0.7333 & 0.1315 \\ -0.0319 & -0.0620 \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix}$$

consider the actuator dynamics

$$\delta_a(s) = \frac{20.2}{s + 20.2} u_a \quad \text{or} \quad \dot{\delta}_a = -20.2\delta_a + 20.2u_a$$

$$\delta_r(s) = \frac{20.2}{s + 20.2} u_r \quad \text{or} \quad \dot{\delta}_r = -20.2\delta_r + 20.2u_r$$

this is a tracking problem, where the reference command is r_ϕ (roll angle command) and r_r (yaw rate command); the integral roll angle tracking error becomes

$$\dot{e} = e_\phi = r_\phi - \phi$$

in addition, consider the yaw-rate measurement through a wash-out filter

$$r_w = \frac{s}{s + 1} r$$

define the state

$$\chi_w = \int r_w = \frac{1}{s + 1} r$$

and the yaw-rate tracking error is

$$e_r = r_r - r_w = r_r - r + \chi_w$$

define the states and so on,

$$\underline{\mathbf{x}} = \begin{bmatrix} \beta \\ \phi \\ p \\ r \\ \delta_a \\ \delta_r \\ \chi_w \\ \epsilon \end{bmatrix}, \underline{\mathbf{u}} = \begin{bmatrix} u_a \\ u_r \end{bmatrix}, \underline{\mathbf{r}} = \begin{bmatrix} r_\phi \\ r_r \end{bmatrix}, \underline{\mathbf{y}} = \begin{bmatrix} \epsilon \\ e_r \\ p \\ e_\phi \end{bmatrix}, \underline{\mathbf{z}} = \begin{bmatrix} \epsilon \\ r_w \end{bmatrix}$$

Augmented system

$$\begin{bmatrix} \dot{\beta} \\ \dot{\phi} \\ \dot{p} \\ \dot{r} \\ \dot{\delta}_a \\ \dot{\delta}_r \\ \dot{\chi}_w \\ \dot{\epsilon} \end{bmatrix} = [\mathbf{A}] \begin{bmatrix} \beta \\ \phi \\ p \\ r \\ \delta_a \\ \delta_r \\ \chi_w \\ \epsilon \end{bmatrix} + [\mathbf{B}] \begin{bmatrix} u_a \\ u_r \end{bmatrix} + [\mathbf{G}] \begin{bmatrix} r_\phi \\ r_r \end{bmatrix}$$

$$\begin{bmatrix} \epsilon \\ e_r \\ p \\ e_\phi \end{bmatrix} = [\mathbf{C}] \begin{bmatrix} \beta \\ \phi \\ p \\ r \\ \delta_a \\ \delta_r \\ \chi_w \\ \epsilon \end{bmatrix} + [\mathbf{F}] \begin{bmatrix} r_\phi \\ r_r \end{bmatrix}$$

$$\begin{bmatrix} \epsilon \\ r_w \end{bmatrix} = [\mathbf{H}] \begin{bmatrix} \beta \\ \phi \\ p \\ r \\ \delta_a \\ \delta_r \\ \chi_w \\ \epsilon \end{bmatrix}$$

leading to, in s-domain

$$\begin{bmatrix} \epsilon \\ r_w \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} u_a \\ u_r \end{bmatrix} + [G_0] \begin{bmatrix} r_\phi \\ r_r \end{bmatrix}$$

Bode Plots of each transfer function

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Singular value

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3. Robustness Performance Specifications

At each frequency $\mathbf{G}(j\omega)$ has singular values $\sigma_1 \geq \dots \geq \sigma_r$, and

$$\underline{\sigma}(\mathbf{G}(j\omega)) = \sigma_r, \quad \text{the smallest}$$

$$\bar{\sigma}(\mathbf{G}(j\omega)) = \sigma_1, \quad \text{the largest}$$

Both remain functions of ω : the extrema are taken over the *index*, not over frequency. Some useful properties:

$$\underline{\sigma}(\mathbf{GK}) - 1 \leq \underline{\sigma}(\mathbf{1} + \mathbf{GK}) \leq \underline{\sigma}(\mathbf{GK}) + 1$$

$$\bar{\sigma}(\mathbf{M}) = \frac{1}{\underline{\sigma}(\mathbf{M}^{-1})}$$

$$\bar{\sigma}(\mathbf{AB}) \leq \bar{\sigma}(\mathbf{A})\bar{\sigma}(\mathbf{B})$$

$\mathbf{S}(j\omega)$ small at low frequency

$$\bar{\sigma}(\mathbf{S}) = \bar{\sigma}[(\mathbf{1} + \mathbf{GK})^{-1}] = \frac{1}{\underline{\sigma}(\mathbf{1} + \mathbf{GK})} \approx \frac{1}{\underline{\sigma}(\mathbf{GK})}$$

large value of $\underline{\sigma}(\mathbf{GK})$;

$\mathbf{T}(j\omega)$ small at high frequency

$$\bar{\sigma}(\mathbf{T}) = \bar{\sigma}[\mathbf{GK}(\mathbf{1} + \mathbf{GK})^{-1}] \approx \bar{\sigma}(\mathbf{GK})$$

small value of $\bar{\sigma}(\mathbf{GK})$;

desirable specs in plot

- ▶ $\underline{\sigma}(\mathbf{GK})$ large at low-frequency;
- ▶ $\bar{\sigma}(\mathbf{GK})$ small at high-frequency

Model Uncertainty

The Loop gain shaping is also valuable to deal with the stability robustness in the presence of model uncertainties. Assume the plant model $\mathbf{G}_\Delta$ now represents a set of models including uncertainties:

$$\Pi = \{\mathbf{G} + \mathbf{W}_1 \Delta \mathbf{W}_2 : \Delta\}$$

and $\mathbf{K}$ stabilizes the nominal plant $\mathbf{G}$, then the closed-loop system is internally stable for all $\|\Delta\|_\infty \leq 1$ if and only if $\|\mathbf{W}_2 \mathbf{K} \mathbf{S} \mathbf{W}_1\|_\infty \leq 1$ (a theorem from [Zhou1996] (ch9) without proof here).

Alternatively, if the uncertainty is represented by

$$\Pi = \{(1 + W_1 \Delta W_2)G : \Delta\}$$

and K stabilizes the nominal plant G , then the closed-loop system is internally stable for all $\|\Delta\|_\infty \leq 1$ if and only if $\|W_2 T W_1\|_\infty \leq 1$ (a theorem from [Zhou1996] (ch9) without proof here). Both results connect the sensitivity (co-sensitivity) matrix to the robustness specifications, that is associated with the loop gain $L = GK$.

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Low-frequency specifications

suppose $n = 0$, $e = \mathbf{S}(r - d)$

1. to keep $\mathbf{S}$ small, as to keep $\bar{\sigma}(\mathbf{S})$ small, leading to

$$\underline{\sigma}(\mathbf{GK}(j\omega)) \gg 1, \quad \omega \leq \omega_d$$

2. steady-state error consideration, suppose $d = 0$, $r = 1/sr_0$
step input

$$e_{ss} = \lim_{s \rightarrow 0} se(s) = \mathbf{S}(0)r_0 \leq \delta_\infty$$

acceptable ss.e value, leading to

$$\underline{\sigma}(\mathbf{GK}(0)) > r_0/\delta_\infty$$

High-frequency specifications

suppose $d = 0$, $e = \mathbf{S}r + \mathbf{T}n$

1. to keep $\mathbf{T}$ small, as to keep $\bar{\sigma}(\mathbf{T})$ small, leading to

$$\bar{\sigma}(\mathbf{G}\mathbf{K}(j\omega)) \ll 1, \quad \omega \geq \omega_n$$

2. to consider unmodeled high-frequency dynamics modes

$$\mathbf{G}_{\Delta}(j\omega) = [\mathbf{1} + \mathbf{M}(j\omega)]\mathbf{G}(j\omega)$$

where $\mathbf{G}$ is the nominal model and the unknown discrepancy satisfies

$$\bar{\sigma}(\mathbf{M}(j\omega)) \leq m(\omega)$$

leading to

$$\bar{\sigma}(\mathbf{G}\mathbf{K}(j\omega)) < \frac{1}{m(\omega)}$$

4. Robust Output Feedback Design

Design Procedure

1. augment the plant with added dynamic terms to achieve the required steady-state behaviour
2. select performance index, with $\mathbf{Q}, \mathbf{R}$
3. design control $\mathbf{K}$ using linear quadratic methods
4. verify the performance satisfaction by simulation
5. determine the low-frequency and high-frequency bounds for stability robustness and performance robustness

[Example 2] Consider the short-period flight dynamics for pitch-rate (tracking) control, and limit the control for δ_e only ([stevens2003], Section 6.3)

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} -1.01887 & 0.90506 \\ 0.82225 & -1.07741 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} -0.00215 \\ -0.17555 \end{bmatrix} \delta_e$$

Consider the actuator dynamics

$$\delta_e(s) = \frac{20.2}{s + 20.2} u_e$$

and AoA sensor measurement:

$$\alpha_F(s) = \frac{10}{s + 10} \alpha$$

Further, $e = r - z = r - q$ and $\epsilon = \int e$

By taking into account of the actuation and AoA states,

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\delta}_e \\ \dot{\alpha}_F \\ \dot{\epsilon} \end{bmatrix} = \begin{bmatrix} -1.01887 & 0.90506 & -0.00215 & 0 & 0 \\ 0.82225 & -1.07741 & -0.17555 & 0 & 0 \\ 0 & 0 & -20.2 & 0 & 0 \\ 10 & 0 & 0 & -10 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \delta_e \\ \alpha_F \\ \epsilon \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 20.2 \\ 0 \\ 0 \end{bmatrix} u_e + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$\begin{bmatrix} \alpha_F \\ q \\ \epsilon \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \delta_e \\ \alpha_F \\ \epsilon \end{bmatrix}$$

$$z = q = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \delta_e \\ \alpha_F \\ \epsilon \end{bmatrix}$$

Select

$$\mathbf{Q} = \mathbf{1}$$

$$R = 1$$

design output feedback control using LQR

$$u_e = -\mathbf{K}\mathbf{y} = - \begin{bmatrix} -0.0004 & -1.9411 & 1.0000 \end{bmatrix} \begin{bmatrix} \alpha_F \\ q \\ \epsilon \end{bmatrix}$$

consider wind gust

$$\phi_w(\omega) = 2L\sigma^2 \frac{1 + 3L^2\omega^2}{(1 + L^2\omega^2)^2}$$

where we choose $L = 3.39, \sigma = 0.1$

consider model uncertainty

$$\mathbf{G}(s) = \mathbf{G}_0(s)[\mathbf{1} + \mathbf{M}(s)]$$

where

$$M(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} - 1$$

and we choose $\omega_n = 40, \xi = 0.3$

closed-loop response: $K_1 = K, K_2 = 10K_1$

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robustness of $K_1, K_2 = 10K_1$

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Summary

- 1 Robustness
- 2 Multivariable Frequency-domain Analysis
- 3 Robustness Performance Specifications
- 4 Robust Output Feedback Design

Summary: What It is About

- ▶ stability robustness and performance robustness
 - ▶ represented by singular values vs frequency
 - ▶ sensitivity and co-sensitivity matrix
 - ▶ singular values of the loop gain **GK**

Further Readings

- ▶ stability robustness and performance robustness [stevens2003] (ch6.2)
- ▶ robust output feedback control [stevens2003] (ch6.3)

References