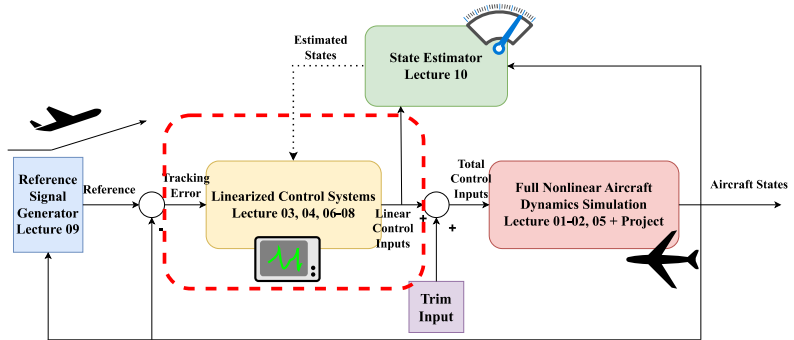


Lecture 07

Linear Quadratic Regulator and Tracking (LQR and LQT)

November 4, 2026

Acknowledgement: These slides were developed with reference to course materials by Professor Hugh H. T. Liu.



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If $(\mathbf{A}, \mathbf{B})$ is stabilizable and $(\mathbf{A}, \mathbf{Q}^{1/2})$ is detectable, then

- ▶ the Riccati equation has a unique solution $\mathbf{P} = \mathbf{P}^T \geq 0$;
- ▶ the closed loop under $\underline{u} = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\underline{x}$ is asymptotically stable.

Both hypotheses are needed. Stabilizability alone leaves an unstable mode that $\mathbf{Q}$ cannot see, and no choice of weights will penalise it. If $(\mathbf{A}, \mathbf{Q}^{1/2})$ is observable rather than merely detectable, then $\mathbf{P} > 0$.

The PI

$$\begin{aligned} J &= \frac{1}{2} \int_0^{\infty} \left(\underline{x}^T \mathbf{Q} \underline{x} + \underline{u}^T \mathbf{R} \underline{u} \right) dt \\ &= -\frac{1}{2} \int_0^{\infty} \frac{d}{dt} (\underline{x}^T \mathbf{P} \underline{x}) dt \\ &= \frac{1}{2} \underline{x}_0^T \mathbf{P} \underline{x}_0 - \frac{1}{2} \lim_{t \rightarrow \infty} \underline{x}^T \mathbf{P} \underline{x} \end{aligned}$$

when the closed-loop system is asymptotically stable, $\underline{x} \rightarrow 0$,

$$J_{\min} = J^* = \frac{1}{2} \underline{x}_0^T \mathbf{P} \underline{x}_0$$

Design Procedure

1. select $\mathbf{Q}, \mathbf{R}$
2. solve the Riccati equation for $\mathbf{P}$
3. controller: $\underline{\mathbf{u}} = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\underline{\mathbf{x}}$

[Example 1] Consider the same longitudinal pitch pointing flight control case study that we discussed in the previous chapter, in an augmented state-space expression format:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\gamma} \\ \dot{\delta}_e \\ \dot{\delta}_f \end{bmatrix} = \begin{bmatrix} -1.341 & 0.9933 & 0 & -0.1689 & -0.2518 \\ 43.223 & -0.8693 & 0 & -17.251 & -1.5766 \\ 1.341 & 0.0067 & 0 & 0.1689 & 0.2518 \\ 0 & 0 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & -20 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \gamma \\ \delta_e \\ \delta_f \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 20 & 0 \\ 0 & 20 \end{bmatrix} \begin{bmatrix} u_e \\ u_f \end{bmatrix}$$

Step 1) to select $\mathbf{Q}$, $\mathbf{R}$ where we choose the simple diagonal matrices to satisfy the symmetric property, with tuning parameters as follows

$$\mathbf{Q} = 0.5 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \\ & & & & 0 \end{bmatrix}$$

$$\mathbf{R} = 0.1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where the values 0.5, 0.1 can be adjusted. Please note the two 0s in the matrix $\mathbf{Q}$ elements as they are associated with the actuator states δ_e, δ_f that are not of the main concern here, and it makes $\mathbf{Q} \geq 0$ still valid.

Step 2) to solve the Riccati equation by using numerical tools (e.g. `lqr` from Matlab©), we obtain the symmetric matrix

$$\mathbf{P} = \begin{bmatrix} 0.7834 & 0.0767 & 0.5449 & -0.0400 & -0.0052 \\ 0.0767 & 0.0305 & 0.0312 & -0.0122 & -0.0013 \\ 0.5449 & 0.0312 & 0.8041 & -0.0109 & 0.0024 \\ -0.0400 & -0.0122 & -0.0109 & 0.0065 & 0.0008 \\ -0.0052 & -0.0013 & 0.0024 & 0.0008 & 0.0001 \end{bmatrix}$$

followed by Step 3) to calculate the state-feedback control law

$$\mathbf{K} = \begin{bmatrix} -7.9914 & -2.4352 & -2.1845 & 1.2970 & 0.1511 \\ -1.0350 & -0.2604 & 0.4774 & 0.1511 & 0.0278 \end{bmatrix}$$

that can be confirmed that $\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}$. Finally, the designed control is implemented, that the closed-loop system response to the initial condition is illustrated here.

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figures/stevens5_2_3lqr_fig1

LQR State-Feedback Closed-loop System Response to Initial Condition $[0.2 \ 0 \ 0 \ 0 \ 0]^T$

2. LQR With Output Feedback

Consider the state-space dynamics:

$$\begin{cases} \dot{\underline{x}} = \underline{A}\underline{x} + \underline{B}\underline{u} \\ \underline{y} = \underline{C}\underline{x} \\ \underline{u} = -\underline{K}\underline{y} = -\underline{K}\underline{C}\underline{x} \end{cases}$$

Linear Quadratic Performance Index:

$$J = \frac{1}{2} \int_0^{\infty} \left(\underline{x}^T \underline{Q} \underline{x} + \underline{u}^T \underline{R} \underline{u} \right) dt$$

where $\underline{Q} = \underline{Q}^T \geq 0$, $\underline{R} = \underline{R}^T > 0$

The closed-loop system:

$$\mathbf{A}_c = \mathbf{A} - \mathbf{B}\mathbf{K}\mathbf{C}$$

Assume there exists $\mathbf{P} = \mathbf{P}^T \geq 0$ s.t.

$$\frac{d}{dt}(\underline{x}^T \mathbf{P} \underline{x}) = -\underline{x}^T (\mathbf{Q} + (\mathbf{K}\mathbf{C})^T \mathbf{R} (\mathbf{K}\mathbf{C})) \underline{x}$$

on the other hand,

$$\begin{aligned} \frac{d}{dt}(\underline{x}^T \mathbf{P} \underline{x}) &= \dot{\underline{x}}^T \mathbf{P} \underline{x} + \underline{x}^T \mathbf{P} \dot{\underline{x}} \\ &= \underline{x}^T (\mathbf{A}_c^T \mathbf{P} + \mathbf{P} \mathbf{A}_c) \underline{x} \\ &= -\underline{x}^T (\mathbf{Q} + (\mathbf{K}\mathbf{C})^T \mathbf{R} (\mathbf{K}\mathbf{C})) \underline{x} \end{aligned}$$

leading to the Lyapunov equation for the output-feedback loop

$$\mathbf{A}^T \mathbf{P} - (\mathbf{K}\mathbf{C})^T \mathbf{B}^T \mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{K}\mathbf{C} + (\mathbf{K}\mathbf{C})^T \mathbf{R}(\mathbf{K}\mathbf{C}) + \mathbf{Q} = 0$$

Why the Standard Riccati Form Fails Here

For *state* feedback the equation collapses to

$A^T P + PA - PBR^{-1}B^T P + Q = 0$ with $K = R^{-1}B^T P$. That step is not available here.

It would require $KC = R^{-1}B^T P$. But K is $m \times p$ and C is $p \times n$, so KC lies in the p -dimensional row space of C , while $R^{-1}B^T P$ is a general $m \times n$ matrix. With $p < n$ no such K exists in general: for the example below the closest K still leaves a 12% residual, in the actuator states C cannot see.

Output-feedback LQR is a harder problem than state-feedback LQR. The equation above couples P and K ; they must be solved together by iteration, not in closed form. Only the *structure* of the design procedure carries over.

Design Procedure

1. select $\mathbf{Q}, \mathbf{R}$;
2. solve the *state-feedback* Riccati equation for $\mathbf{P}$;
3. take the $\mathbf{K}$ that minimises $\|\mathbf{K}\mathbf{C} - \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\|_F$, and apply $\underline{\mathbf{u}} = -\mathbf{K}\underline{\mathbf{y}}$;
4. **verify** that $\mathbf{A} - \mathbf{B}\mathbf{K}\mathbf{C}$ is stable.

Step 3 is a projection onto the row space of $\mathbf{C}$, not a solution: it leaves a residual by construction. Step 4 is therefore not optional — it is the only thing that certifies the design.

[Example 2] Consider the augmented short-period flight dynamics [stevens2003] (ch5),

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\gamma} \\ \dot{\delta}_e \\ \dot{\delta}_f \end{bmatrix} = \begin{bmatrix} -1.341 & 0.9933 & 0 & -0.1689 & -0.2518 \\ 43.223 & -0.8693 & 0 & -17.251 & -1.5766 \\ 1.341 & 0.0067 & 0 & 0.1689 & 0.2518 \\ 0 & 0 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & -20 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \gamma \\ \delta_e \\ \delta_f \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 20 & 0 \\ 0 & 20 \end{bmatrix} \begin{bmatrix} u_e \\ u_f \end{bmatrix}$$

The output

and the output is identified by $\underline{y}^T = [q \ n_z \ \gamma]$,

$$\begin{bmatrix} q \\ n_z \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 47.76 & -0.268 & 0 & -4.56 & 4.45 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \gamma \\ \delta_e \\ \delta_f \end{bmatrix}$$

Step 1) to select the same $\mathbf{Q}, \mathbf{R}$ as the one we used in state-feedback case,

$$\mathbf{Q} = 0.5 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \\ & & & & 0 \end{bmatrix}$$

$$\mathbf{R} = 0.1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Step 2) to solve the Riccati equation we will get the identical result

$$\mathbf{P} = \begin{bmatrix} 0.7834 & 0.0767 & 0.5449 & -0.0400 & -0.0052 \\ 0.0767 & 0.0305 & 0.0312 & -0.0122 & -0.0013 \\ 0.5449 & 0.0312 & 0.8041 & -0.0109 & 0.0024 \\ -0.0400 & -0.0122 & -0.0109 & 0.0065 & 0.0008 \\ -0.0052 & -0.0013 & 0.0024 & 0.0008 & 0.0001 \end{bmatrix}$$

followed by Step 3) to calculate the output-feedback control law

$$\mathbf{K} = \begin{bmatrix} -2.4798 & -0.1667 & -2.1845 \\ -0.2662 & -0.0215 & 0.4774 \end{bmatrix}$$

the least-squares solution of $\mathbf{KC} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}$. It leaves a 12% residual, almost all of it in the δ_e and δ_f columns that $\mathbf{C}$ cannot see; the closed loop $\mathbf{A} - \mathbf{BKC}$ is nevertheless stable, which is what makes the gain usable.

[missing figure]
figures/stevens5_2_3lqr_fig2

LQR Output-Feedback Closed-loop System Response to Initial Condition $[0.2 \ 0 \ 0 \ 0 \ 0]^T$

[Example 3] Consider the augmented lateral flight dynamics [stevens2003] (ch5),

$$\begin{bmatrix} \dot{\beta} \\ \dot{\phi} \\ \dot{p} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} -0.3220 & 0.0640 & 0.0364 & -0.9917 \\ 0 & 0 & 1 & 0.0037 \\ -30.6492 & 0 & -3.6784 & 0.6646 \\ 8.5396 & 0 & -0.0254 & -0.4764 \end{bmatrix} \begin{bmatrix} \beta \\ \phi \\ p \\ r \end{bmatrix} + \begin{bmatrix} -0.9917 & 0.0003 \\ 0 & 0 \\ -0.7333 & 0.1315 \\ -0.0319 & -0.0620 \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix}$$

consider the actuator dynamics

$$\delta_a(s) = \frac{20.2}{s + 20.2} u_a \quad \text{or} \quad \dot{\delta}_a = -20.2\delta_a + 20.2u_a$$
$$\delta_r(s) = \frac{20.2}{s + 20.2} u_r \quad \text{or} \quad \dot{\delta}_r = -20.2\delta_r + 20.2u_r$$

in addition, consider the wash-out filter

$$r_w = \frac{s}{s+1} r, \quad \text{or} \quad \dot{r}_w = \dot{r} - r_w$$

define the state

$$\chi_w = \int r_w$$

$$\dot{\chi}_w = r_w = r - \chi_w$$

Augmented system

$$\begin{bmatrix} \dot{\beta} \\ \dot{\phi} \\ \dot{p} \\ \dot{r} \\ \dot{\delta}_a \\ \dot{\delta}_r \\ \dot{\chi}_w \end{bmatrix} = [A] \begin{bmatrix} \beta \\ \phi \\ p \\ r \\ \delta_a \\ \delta_r \\ \chi_w \end{bmatrix} + [B] \begin{bmatrix} u_a \\ u_r \end{bmatrix}$$

$$\begin{bmatrix} r_w \\ p \\ \beta \\ \phi \end{bmatrix} = [C] \begin{bmatrix} \beta \\ \phi \\ p \\ r \\ \delta_a \\ \delta_r \\ \chi_w \end{bmatrix}$$

Design Procedure

Select

$$\mathbf{Q} = \begin{bmatrix} 50 & & & & & & \\ & 100 & & & & & \\ & & 100 & & & & \\ & & & 50 & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & & & & 0 \end{bmatrix}$$

$$\mathbf{R} = 0.1 \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$$

output feedback

$P =$

$$\begin{bmatrix} 261.5145 & -12.2219 & -9.2747 & -3.6311 & 0.3003 & -0.0701 & -0.0000 \\ -12.2219 & 108.6814 & 7.2033 & 0.5808 & -0.1530 & 0.0267 & -0.0000 \\ -9.2747 & 7.2033 & 6.3687 & -0.0284 & -0.1258 & 0.0235 & 0.0000 \\ -3.6311 & 0.5808 & -0.0284 & 27.4465 & -0.0404 & -0.0773 & 0.0000 \\ 0.3003 & -0.1530 & -0.1258 & -0.0404 & 0.0034 & -0.0005 & -0.0000 \\ -0.0701 & 0.0267 & 0.0235 & -0.0773 & -0.0005 & 0.0003 & -0.0000 \\ -0.0000 & -0.0000 & 0.0000 & 0.0000 & -0.0000 & -0.0000 & 0 \end{bmatrix}$$

from $KC = R^{-1}B^T P$, obtains

$$K = \begin{bmatrix} -4.0789 & -25.4214 & 60.6684 & -30.9080 \\ -7.8043 & 4.7483 & -14.1528 & 5.3901 \end{bmatrix}$$

initial response

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figures/stevens5_3_1lqr_fig1

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figures/stevens5_3_1lqr_fig2

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figures/stevens5_3_1lqr_fig3

3. Tracking Control Problem

When the output of a system is required to follow (track) a desired command.

1. if the plant output is to follow a desired trajectory (path), the problem is referred to as a *servo problem*;
2. if the desired signal is a prescribed function of time, the problem is called a *tracking problem*;
3. if the desired command is represented by an ideal response to a given plant (model), the problem is addressed as the *model-following problem*

[missing figure]
figures/trackcontrol

- ▶ r command (reference)
- ▶ u control input
- ▶ z performance (to track command)
- ▶ y output

4. Linear Quadratic Tracking Control

Consider the state-space dynamics:

$$\begin{cases} \dot{\underline{x}} = \underline{A}\underline{x} + \underline{B}\underline{u} + \underline{E}\underline{r} \\ \underline{y} = \underline{C}\underline{x} + \underline{F}\underline{r} \\ \underline{z} = \underline{H}\underline{x} \\ \underline{u} = -\underline{K}\underline{y} \end{cases}$$

The closed-loop system:

$$\dot{\underline{x}} = (\underline{A} - \underline{B}\underline{K}\underline{C})\underline{x} + (\underline{E} - \underline{B}\underline{K}\underline{F})\underline{r} = \underline{A}_c\underline{x} + \underline{B}_c\underline{r}$$

steady states, output and performance:

$$\begin{cases} \underline{x}_{ss} &= -\underline{A}_c^{-1}\underline{B}_c\underline{r} \\ \underline{y}_{ss} &= \underline{C}\underline{x}_{ss} + \underline{F}\underline{r} \\ \underline{z}_{ss} &= \underline{H}\underline{x}_{ss} \end{cases}$$

define the deviation states:

$$\begin{cases} \tilde{\underline{x}} &= \underline{x} - \underline{x}_{ss} \\ \tilde{\underline{y}} &= \underline{y} - \underline{y}_{ss} = \underline{C}\underline{x} + \underline{F}\underline{r} - \underline{C}\underline{x}_{ss} - \underline{F}\underline{r} = \underline{C}\tilde{\underline{x}} \\ \tilde{\underline{z}} &= \underline{z} - \underline{z}_{ss} = \underline{H}\underline{x} - \underline{H}\underline{x}_{ss} = \underline{H}\tilde{\underline{x}} \end{cases}$$

Tracking error:

$$\underline{e} = \underline{r} - \underline{z}$$

therefore

$$\underline{\tilde{e}} = \underline{e} - \underline{e}_{ss} = (\underline{r} - \underline{z}) - (\underline{r} - \underline{z}_{ss}) = -\underline{\tilde{z}} = -\underline{H}\underline{\tilde{x}}$$

The Deviation System

Take the command $\underline{r}$ constant, so that $\underline{x}_{ss}$ is fixed and $\dot{\underline{x}}_{ss} = \mathbf{0}$. The steady-state terms then cancel:

$$\begin{aligned}
 \underline{\tilde{u}} &= \underline{u} - \underline{u}_{ss} \\
 &= -K C \underline{x} - K F \underline{r} - \left(-K C \underline{x}_{ss} - K F \underline{r} \right) \\
 &= -K C (\underline{x} - \underline{x}_{ss}) = -K C \underline{\tilde{x}} \\
 \underline{\dot{\tilde{x}}} &= \underline{\dot{x}} - \underline{\dot{x}}_{ss} \\
 &= \underline{A_c x} + \underline{B_c r} - (\underline{A_c x}_{ss} + \underline{B_c r}) \\
 &= \underline{A_c \tilde{x}} \\
 &= (\underline{A} - \underline{B K C}) \underline{\tilde{x}} \\
 &= \underline{A \tilde{x}} + \underline{B \tilde{u}}
 \end{aligned}$$

leading to

$$\begin{cases} \dot{\underline{\tilde{x}}} &= \mathbf{A}\underline{\tilde{x}} + \mathbf{B}\underline{\tilde{u}} \\ \underline{\tilde{y}} &= \mathbf{C}\underline{\tilde{x}} \\ \underline{\tilde{z}} &= \mathbf{H}\underline{\tilde{x}} = -\underline{\tilde{e}} \\ \underline{\tilde{u}} &= -\mathbf{K}\underline{\tilde{y}} \end{cases}$$

Linear Quadratic Performance Index:

$$\begin{aligned} J &= \frac{1}{2} \int_0^{\infty} \left(\underline{\tilde{x}}^T \mathbf{Q} \underline{\tilde{x}} + \underline{\tilde{u}}^T \mathbf{R} \underline{\tilde{u}} \right) dt \\ &= \frac{1}{2} \int_0^{\infty} \left(\underline{\tilde{e}}^T \underline{\tilde{e}} + \underline{\tilde{u}}^T \mathbf{R} \underline{\tilde{u}} \right) dt \end{aligned}$$

where $\mathbf{Q} = \mathbf{H}^T \mathbf{H} = \mathbf{Q}^T \geq 0$, $\mathbf{R} = \mathbf{R}^T > 0$.

The LQR problem formulation for the deviation system leads to $\underline{\tilde{x}} \rightarrow 0$, $\underline{\tilde{e}} \rightarrow 0$, then the tracking error:

$$\underline{e} \rightarrow \underline{e}_{ss}$$

or $\underline{z}$ follows $\underline{r}$ with a steady-state error.

The optimal (minimum) tracking performance index becomes
(leave as an exercise)

$$J^* = \frac{1}{2} \tilde{\underline{x}}_0^T \mathbf{P} \tilde{\underline{x}}_0$$

where $\mathbf{P}$ is the solution of the Riccati equation, and $\tilde{\underline{x}}_0 = \underline{x}_0 - \underline{x}_{ss}$.

LQ Tracking Design Procedure

Generally speaking, the linear quadratic tracking control follows the performance index:

$$J = \frac{1}{2} \int_0^{\infty} \left(\underline{\mathbf{e}}^T \mathbf{Q} \underline{\mathbf{e}} + \underline{\mathbf{u}}^T \mathbf{R} \underline{\mathbf{u}} \right) dt$$

to design the control (either state feedback or output feedback)

[Example 4] Consider the short-period flight dynamics for normal acceleration tracking, and limit the control for δ_e only ([stevens2003], Section 5.4)

$$\begin{aligned} \begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} &= \begin{bmatrix} -1.01887 & 0.90506 \\ 0.82225 & -1.07741 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} \\ &\quad + \begin{bmatrix} -0.00215 \\ -0.17555 \end{bmatrix} \delta_e \\ n_z &= \begin{bmatrix} 16.26 & 0.9788 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} - 0.04852 \delta_e \end{aligned}$$

Consider the actuator dynamics

$$\delta_e(s) = \frac{20.2}{s + 20.2} u_e$$

and AoA sensor measurement:

$$\alpha_F(s) = \frac{10}{s + 10} \alpha$$

Further, $e = r - z = r - n_z$ and $\epsilon = \int e$

Linear Quadratic Tracking: Deviation system approach

The augmented system:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\delta}_e \\ \dot{\alpha}_F \\ \frac{\dot{\alpha}_F}{\dot{\epsilon}(e)} \end{bmatrix} = \begin{bmatrix} -1.01887 & 0.90506 & -0.00215 & 0 & 0 \\ 0.82225 & -1.07741 & -0.17555 & 0 & 0 \\ 0 & 0 & -20.2 & 0 & 0 \\ 10 & 0 & 0 & -10 & 0 \\ -16.26 & -0.9788 & 0.04852 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \delta_e \\ \alpha_F \\ \frac{\alpha_F}{\epsilon} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 20.2 \\ 0 \\ 0 \end{bmatrix} u_e + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$\underline{y} = \begin{bmatrix} \alpha_F \\ q \\ e \\ \epsilon \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -16.26 & -0.9788 & 0.04852 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \delta_e \\ \alpha_F \\ \epsilon \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} r$$

$$z = \left[\begin{array}{cccc|c} 16.26 & 0.9788 & -0.04852 & 0 & 0 \end{array} \right] \begin{bmatrix} \alpha \\ q \\ \delta_e \\ \frac{\alpha_F}{\epsilon} \end{bmatrix}$$

Consider the performance index:

$$J = \frac{1}{2} \int_0^{\infty} \left(\underline{\tilde{x}}^T \mathbf{Q} \underline{\tilde{x}} + \underline{\tilde{u}}^T \mathbf{R} \underline{\tilde{u}} \right) dt$$

$$\mathbf{Q} = \mathbf{H}^T \mathbf{H} = \begin{bmatrix} 264.3876 & 15.9153 & -0.7889 & 0 & 0 \\ 15.9153 & 0.9580 & -0.0475 & 0 & 0 \\ -0.7889 & -0.0475 & 0.0024 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{R} = 1$$

leading to

$$\begin{aligned} u_e &= -\mathbf{K}\mathbf{y} \\ &= -\begin{bmatrix} 0 & -8.7503 & 1.1733 & 1.0000 \end{bmatrix} \begin{bmatrix} \alpha_F \\ q \\ e \\ \epsilon \end{bmatrix} \end{aligned}$$

[missing figure]
figures/stevens5_4_11q_fig2

Linear Quadratic Integral Design: lqi (Matlab)

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figures/lqi

By taking into account of the actuation and AoA states,

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\delta}_e \\ \dot{\alpha}_F \end{bmatrix} = \begin{bmatrix} -1.01887 & 0.90506 & -0.00215 & 0 \\ 0.82225 & -1.07741 & -0.17555 & 0 \\ 0 & 0 & -20.2 & 0 \\ 10 & 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \delta_e \\ \alpha_F \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 20.2 \\ 0 \end{bmatrix} u_e$$

$$y = \begin{bmatrix} 16.26 & 0.9788 & -0.04852 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \delta_e \\ \alpha_F \end{bmatrix}$$

Select

$$Q = \begin{bmatrix} H^T H & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 264.3876 & 15.9153 & -0.7889 & 0 & 0 \\ 15.9153 & 0.9580 & -0.0475 & 0 & 0 \\ -0.7889 & -0.0475 & 0.0024 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R = 1$$

According to the LQI structure, control design

$$\begin{aligned}
 u_e &= -\underline{\mathbf{K}} \begin{bmatrix} \underline{x} \\ \epsilon(x_i) \end{bmatrix} \\
 &= - \left[\begin{array}{cccc|c} -19.4168 & -9.5000 & 0.0846 & 0 & 1.0000 \end{array} \right] \begin{bmatrix} \alpha \\ q \\ \delta_e \\ \frac{\alpha_F}{\epsilon} \end{bmatrix}
 \end{aligned}$$

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figures/stevens5_4_11q_fig1

5. Model-Following Tracking

to design a control system to make the actual aircraft behave like an ideal model.

5.1 Regulation

Consider the system:

$$\begin{cases} \dot{\underline{x}} = \mathbf{A}\underline{x} + \mathbf{B}\underline{u} \\ \underline{y} = \mathbf{C}\underline{x} \\ \underline{z} = \mathbf{H}\underline{x} \end{cases}$$

The ideal (reference) model is represented by:

$$\begin{cases} \dot{\underline{x}}_m = \mathbf{A}_m \underline{x}_m \\ \underline{y}_m = \mathbf{C}_m \underline{x}_m \\ \underline{z}_m = \mathbf{H}_m \underline{x}_m \end{cases}$$

the design objective is to minimize the model mismatch error:

$$\underline{e} = \underline{z}_m - \underline{z} = \mathbf{H}_m \underline{x}_m - \mathbf{H} \underline{x}$$

the performance index:

$$J = \frac{1}{2} \int_0^{\infty} \left(\underline{e}^T \mathbf{Q} \underline{e} + \underline{u}^T \mathbf{R} \underline{u} \right) dt$$

the augmented system, let

$$\underline{X} = \begin{bmatrix} \underline{x} \\ \underline{x}_m \end{bmatrix}$$

$$\dot{\underline{X}} = \begin{bmatrix} \dot{\underline{x}} \\ \dot{\underline{x}}_m \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \\ & \mathbf{A}_m \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{x}_m \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \underline{u}$$

$$\underline{Y} = \begin{bmatrix} \underline{y} \\ \underline{y}_m \end{bmatrix} = \begin{bmatrix} \mathbf{C} & \\ & \mathbf{C}_m \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{x}_m \end{bmatrix}$$

$$\underline{e} = \begin{bmatrix} -\mathbf{H} & \mathbf{H}_m \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{x}_m \end{bmatrix}$$

and the performance index becomes

$$J = \frac{1}{2} \int_0^{\infty} \left(\underline{X}^T \bar{\mathbf{Q}} \underline{X} + \underline{u}^T \mathbf{R} \underline{u} \right) dt$$

$$\begin{aligned}
 \underline{e}^T Q \underline{e} &= \left(\begin{bmatrix} -H & H_m \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{x}_m \end{bmatrix} \right)^T Q \begin{bmatrix} -H & H_m \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{x}_m \end{bmatrix} \\
 &= \underline{x}^T \begin{bmatrix} -H^T \\ H_m^T \end{bmatrix} Q \begin{bmatrix} -H & H_m \end{bmatrix} \underline{x}
 \end{aligned}$$

and

$$\bar{Q} = \begin{bmatrix} H^T Q H & -H^T Q H_m \\ -H_m^T Q H & H_m^T Q H_m \end{bmatrix}$$

design LQR control

$$\underline{u} = -K \underline{Y}$$

5.2 Tracking

Consider the system:

$$\begin{cases} \dot{\underline{x}} = \underline{A}\underline{x} + \underline{B}\underline{u} + \underline{E}\underline{r} \\ \underline{y} = \underline{C}\underline{x} + \underline{F}\underline{r} \\ \underline{z} = \underline{H}\underline{x} \end{cases}$$

The ideal (reference) model is represented by:

$$\begin{cases} \dot{\underline{x}}_m = \underline{A}_m\underline{x}_m + \underline{E}_m\underline{r} \\ \underline{y}_m = \underline{C}_m\underline{x}_m + \underline{F}_m\underline{r} \\ \underline{z}_m = \underline{H}_m\underline{x}_m \end{cases}$$

the design objective is to minimize the model mismatch error:

$$\underline{e} = \underline{z}_m - \underline{z} = \mathbf{H}_m \underline{x}_m - \mathbf{H} \underline{x}$$

the augmented system, let

$$\underline{\mathbf{X}} = \begin{bmatrix} \underline{\mathbf{x}} \\ \underline{\mathbf{x}}_m \end{bmatrix}$$

$$\dot{\underline{\mathbf{X}}} = \begin{bmatrix} \mathbf{A} & \mathbf{A}_m \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{x}} \\ \underline{\mathbf{x}}_m \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \underline{\mathbf{u}} + \begin{bmatrix} \mathbf{E} \\ \mathbf{E}_m \end{bmatrix} \underline{\mathbf{r}}$$

$$\underline{\mathbf{Y}} = \begin{bmatrix} \mathbf{C} & \mathbf{C}_m \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{x}} \\ \underline{\mathbf{x}}_m \end{bmatrix} + \begin{bmatrix} \mathbf{F} \\ \mathbf{F}_m \end{bmatrix} \underline{\mathbf{r}}$$

$$\underline{\mathbf{e}} = \begin{bmatrix} -\mathbf{H} & \mathbf{H}_m \end{bmatrix} \begin{bmatrix} \underline{\mathbf{x}} \\ \underline{\mathbf{x}}_m \end{bmatrix}$$

Consider the state-space dynamics:

$$\dot{\underline{x}} = \bar{\mathbf{A}}\underline{x} + \bar{\mathbf{B}}\underline{u} + \bar{\mathbf{E}}\underline{r}$$

$$\underline{y} = \bar{\mathbf{C}}\underline{x} + \bar{\mathbf{F}}\underline{r}$$

$$\underline{e} = \begin{bmatrix} -\mathbf{H} & \mathbf{H}_m \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{x}_m \end{bmatrix} = \bar{\mathbf{H}}\underline{x}$$

$$\underline{u} = -\bar{\mathbf{K}}\underline{y}$$

The closed-loop system:

$$\dot{\underline{X}} = (\bar{\mathbf{A}} - \bar{\mathbf{B}}\bar{\mathbf{K}}\bar{\mathbf{C}})\underline{X} + (\bar{\mathbf{E}} - \bar{\mathbf{B}}\bar{\mathbf{K}}\bar{\mathbf{F}})\underline{r} = \bar{\mathbf{A}}_c\underline{X} + \bar{\mathbf{B}}_c\underline{r}$$

steady states, output and performance:

$$\begin{cases} \underline{X}_{ss} &= -(\bar{\mathbf{A}}_c)^{-1}\bar{\mathbf{B}}_c\underline{r} \\ \underline{Y}_{ss} &= \bar{\mathbf{C}}\underline{X}_{ss} + \bar{\mathbf{F}}\underline{r} \\ \underline{e}_{ss} &= \bar{\mathbf{H}}\underline{X}_{ss} \end{cases}$$

define the deviation states:

$$\begin{cases} \tilde{\underline{X}} &= \underline{X} - \underline{X}_{ss} \\ \tilde{\underline{Y}} &= \underline{Y} - \underline{Y}_{ss} = \bar{\mathbf{C}}\underline{X} + \bar{\mathbf{F}}\underline{r} - \bar{\mathbf{C}}\underline{X}_{ss} - \bar{\mathbf{F}}\underline{r} = \bar{\mathbf{C}}\tilde{\underline{X}} \\ \tilde{\underline{e}} &= \underline{e} - \underline{e}_{ss} = \bar{\mathbf{H}}\underline{X} - \bar{\mathbf{H}}\underline{X}_{ss} = \bar{\mathbf{H}}\tilde{\underline{X}} \end{cases}$$

The Deviation System

$$\begin{aligned}
 \underline{\tilde{u}} &= \underline{u} - \underline{u}_{ss} \\
 &= -\bar{K}\bar{C}\underline{x} - \bar{K}\bar{F}\underline{r} - \left(-\bar{K}\bar{C}\underline{x}_{ss} - \bar{K}\bar{F}\underline{r} \right) \\
 &= -\bar{K}\bar{C}(\underline{x} - \underline{x}_{ss}) = -\bar{K}\bar{C}\tilde{\underline{x}} \\
 \dot{\tilde{\underline{x}}} &= \dot{\underline{x}} - \dot{\underline{x}}_{ss} \\
 &= \bar{A}_c\underline{x} + \bar{B}_c\underline{r} - (\bar{A}_c\underline{x}_{ss} + \bar{B}_c\underline{r}) \\
 &= \bar{A}_c\tilde{\underline{x}} \\
 &= (\bar{A} - \bar{B}\bar{K}\bar{C})\tilde{\underline{x}} \\
 &= \bar{A}\tilde{\underline{x}} + \bar{B}\underline{\tilde{u}}
 \end{aligned}$$

leading to

$$\begin{cases} \dot{\tilde{\mathbf{X}}} = \bar{\mathbf{A}}\tilde{\mathbf{X}} + \bar{\mathbf{B}}\tilde{\mathbf{u}} \\ \tilde{\mathbf{Y}} = \bar{\mathbf{C}}\tilde{\mathbf{X}} \\ \tilde{\mathbf{u}} = -\bar{\mathbf{K}}\tilde{\mathbf{Y}} \end{cases}$$

Linear Quadratic Performance Index:

$$J = \frac{1}{2} \int_0^{\infty} \left(\tilde{\mathbf{X}}^T \bar{\mathbf{Q}} \tilde{\mathbf{X}} + \tilde{\mathbf{u}}^T \mathbf{R} \tilde{\mathbf{u}} \right) dt$$

[Example 5] an automatic landing system ([stevens2003], Section 5.6)

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\theta} \\ \dot{q} \\ \dot{h} \end{bmatrix} = [A] \begin{bmatrix} \alpha \\ \theta \\ q \\ h \end{bmatrix} + [B]\delta_e$$

where

$$\dot{h} \approx U_e(\theta - \alpha), U_e = 250 \text{ (fps)}$$

to follow a reference landing profile

$$\dot{h}_m = -0.3h_m + r$$

and $r = 0$, $h_0 = 36.67 \text{ (ft)}$.

$$[A] = \begin{bmatrix} -0.64627 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -0.7739 & 0 & -0.52977 & 0 \\ -250 & 250 & 0 & 0 \end{bmatrix}, [B] = \begin{bmatrix} 0 \\ 0 \\ -0.011 \\ 0 \end{bmatrix}$$

taking into account the actuation

$$\delta(s) = \frac{10}{s + 10} u_e$$

tracking error and its integral term:

$$\begin{aligned} e &= h_m - h \\ \epsilon &= \int e \end{aligned}$$

we also assume the measurement of ϵ through a filter

$$\chi(s) = \frac{1}{s + 10} \epsilon$$

the augmented state-space system:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\theta} \\ \dot{q} \\ \dot{h} \\ \dot{\delta_e} \\ \dot{\chi} \\ \dot{\epsilon} \end{bmatrix} = \mathbf{A}_{aug} \begin{bmatrix} \alpha \\ \theta \\ q \\ h \\ \delta_e \\ \chi \\ \epsilon \end{bmatrix} + \mathbf{B}_{aug} u_e + \mathbf{E} h_m$$

assume the output feedback:

$$\begin{bmatrix} \chi \\ \epsilon \\ e \\ q \\ \theta \end{bmatrix} = \mathbf{C}_{aug} \begin{bmatrix} \alpha \\ \theta \\ q \\ h \\ \delta_e \\ \chi \\ \epsilon \end{bmatrix} + \mathbf{D} h_m$$

We can build the tracking system:

$$\dot{\underline{X}} = \begin{bmatrix} \dot{\underline{x}} \\ \dot{h}_m \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{aug} & \mathbf{E} \\ 0 & -0.3 \end{bmatrix} \begin{bmatrix} \underline{x} \\ h_m \end{bmatrix} + \begin{bmatrix} \mathbf{B}_{aug} \\ 0 \end{bmatrix} u_e + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r$$

$$\underline{Y} = \begin{bmatrix} \underline{y} \\ h_m \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{aug} & \mathbf{D} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \underline{x} \\ h_m \end{bmatrix}$$

design control

$$u_e = -\mathbf{K} \underline{Y}$$

Initial response to $h_0 = 36.67 \text{ (ft)}$, $\theta_0 = -2.5 \text{ (deg)}$

[missing figure]
figures/stevens5_6_llq_fig1

[missing figure]
figures/stevens5_6_11q_fig2

Summary

- 1 Review: LQR from Lecture 06
- 2 LQR With Output Feedback
- 3 Tracking Control Problem
- 4 Linear Quadratic Tracking Control
- 5 Model-Following Tracking

References

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