

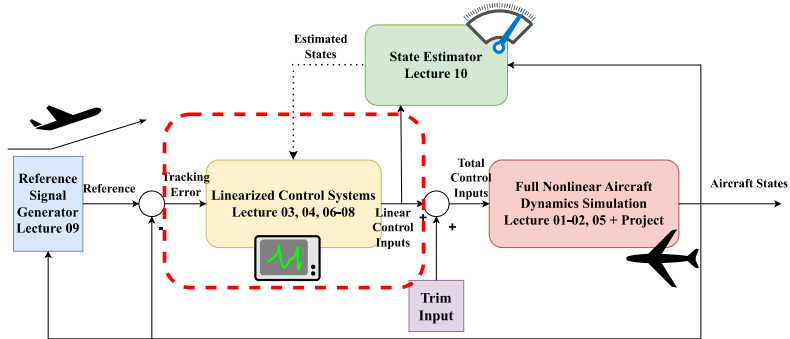
ADVANCED FLIGHT DYNAMICS & CONTROL

Lecture 06

Closed-loop Pole Placement

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1. The State-Space Description

Recall the linearized flight equations of motion presented in previous chapters, e.g., the longitudinal and lateral state-space representations, in the format of:

$$\dot{\underline{x}} = \mathbf{A}_{long}\underline{x} + \mathbf{B}_{long}\underline{u}$$

$$\dot{\underline{x}} = \mathbf{A}_{latr}\underline{x} + \mathbf{B}_{latr}\underline{u}$$

where $\underline{x} = [u \ w \ q \ \theta]^T$ or $\underline{x} = [v \ p \ r \ \phi]^T$ for longitudinal states or lateral states respectively, $\underline{u} = [\delta_e \ \delta_p]^T$ or $\underline{u} = [\delta_a \ \delta_r]^T$ captures the corresponding control inputs accordingly.

For Multivariable, multi-input multi-output (MIMO), we have the following general state-space form:

$$\begin{cases} \dot{\underline{x}}(t) = \mathbf{A}\underline{x}(t) + \mathbf{B}\underline{u}(t) \\ \underline{y}(t) = \mathbf{C}\underline{x}(t) + \mathbf{D}\underline{u}(t) \end{cases}$$

where

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathcal{R}^n, \underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \in \mathcal{R}^m, \underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_r \end{bmatrix} \in \mathcal{R}^r,$$

- A** $n \times n$ system (state) matrix
- B** $n \times m$ input matrix
- C** $r \times n$ output matrix
- D** $r \times m$ direct transmission matrix

Assume we have a full state-feedback control structure,

$$\underline{u} = -\mathbf{K}\underline{x}, \quad \mathbf{K} \in \mathcal{R}^{m \times n}$$

then a closed-loop control system is established,

$$\dot{\underline{x}} = (\mathbf{A} - \mathbf{BK})\underline{x}$$

whose performance is regulated by the control law $\mathbf{K}$.

- ▶ the beauty: MIMO & mathematically systematic
- ▶ the challenge: mathematical

Solutions to the State-Space Equations

Consider the homogeneous system first,

$$\dot{\underline{x}}(t) = \mathbf{A}\underline{x}(t)$$

where $\underline{x}(0)$ is given. The solution becomes

$$\underline{x}(t) = e^{\mathbf{A}t} \underline{x}(0)$$

where

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \mathbf{A}^k \frac{t^k}{k!}$$

If $\mathbf{A} = \mathbf{T}\mathbf{\Lambda}\mathbf{T}^{-1}$, where $\mathbf{T}$ is the transformation matrix, $\mathbf{\Lambda}$ is the diagonal matrix whose elements are eigenvalues of $\mathbf{A}$. Therefore,

$$e^{\mathbf{A}t} = \mathbf{T}e^{\mathbf{\Lambda}t}\mathbf{T}^{-1}$$

where the transient property is determined by the characteristics of the system.

From the Laplace Transform perspective,

$$\mathcal{L} \left\{ e^{\mathbf{A}t} \right\} = [s\mathbf{I} - \mathbf{A}]^{-1}$$

or

$$e^{\mathbf{A}t} = \mathcal{L}^{-1} \left\{ [s\mathbf{I} - \mathbf{A}]^{-1} \right\}$$

[Example 1]

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \text{given} \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$\begin{aligned} [sI - \mathbf{A}]^{-1} &= \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \\ &= \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} \end{aligned}$$

leading to (by modal decomposition)

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$



For Non-homogeneous system

$$\dot{\underline{x}}(t) = \mathbf{A}\underline{x}(t) + \mathbf{B}\underline{u}(t)$$

$$\underline{x}(t) = e^{\mathbf{A}t}\underline{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\underline{u}(\tau) d\tau$$

Laplace Transform

$$\begin{aligned}\underline{x}(s) &= [s\mathbf{1} - \mathbf{A}]^{-1}\underline{x}(0) + [s\mathbf{1} - \mathbf{A}]^{-1}\mathbf{B}\underline{u}(s) \\ &= \mathcal{L}\left\{e^{\mathbf{A}t}\right\}\underline{x}(0) + \mathcal{L}\left\{e^{\mathbf{A}t}\right\}\mathcal{L}\left\{\mathbf{B}\underline{u}(t)\right\}\end{aligned}$$

therefore

$$\underline{x}(t) = e^{\mathbf{A}t}\underline{x}(0) + e^{\mathbf{A}t} * \mathbf{B}\underline{u}(t)$$

Modal Equations and Stability

Eigen problem of Matrix $\mathbf{A}$: there exists a scalar (eigenvalue) λ and its associated vector (eigenvector) $\underline{\mathbf{e}}$ such that:

$$\lambda \underline{\mathbf{e}} = \mathbf{A} \underline{\mathbf{e}}$$

if and only if when

$$\det(\lambda \mathbf{1} - \mathbf{A}) = 0$$

Let $\lambda_i \underline{e}_i = \underline{A} \underline{e}_i$ for $i = 1, \dots, n$, then

$$\begin{bmatrix} \underline{e}_1 & \underline{e}_2 & \dots & \underline{e}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} = \underline{A} \begin{bmatrix} \underline{e}_1 & \underline{e}_2 & \dots & \underline{e}_n \end{bmatrix}$$

or

$$\begin{aligned} \underline{E} \underline{\Lambda} &= \underline{A} \underline{E} \\ \underline{A} &= \underline{E} \underline{\Lambda} \underline{E}^{-1} \\ \underline{\Lambda} &= \underline{E}^{-1} \underline{A} \underline{E} \end{aligned}$$

For homogeneous equation $\dot{\underline{x}}(t) = \mathbf{A}\underline{x}(t)$, let $\underline{x}(t) = \mathbf{E}\underline{q}(t)$,
leading to modal representation

$$\dot{\underline{q}}(t) = \mathbf{\Lambda}\underline{q}(t), \quad \underline{q}(t) = e^{\mathbf{\Lambda}t}\underline{q}(0)$$

and

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{E}e^{\mathbf{\Lambda}t}\mathbf{E}^{-1} \\ \underline{x}(t) &= e^{\mathbf{A}t}\underline{x}(0) = \mathbf{E}e^{\mathbf{\Lambda}t}\mathbf{E}^{-1}\underline{x}(0) \end{aligned}$$

L-Stability

Consider $\dot{\underline{x}} = \mathbf{A}\underline{x}$, whose equilibrium is $\underline{x} = \underline{0}$. It is stable in the sense of Lyapunov, or L-stable, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\underline{x}(0)\| \leq \delta$ implies $\|\underline{x}(t)\| \leq \varepsilon$ for all $t \geq 0$;
asymptotically stable if in addition $\underline{x}(t) \rightarrow \underline{0}$.

- ▶ L-stable iff $\text{Re}\{\lambda_i\} \leq 0$ **and** every λ_i on the imaginary axis is semisimple — no Jordan block larger than 1×1 there
- ▶ asymptotically stable iff $\text{Re}\{\lambda_i\} < 0$

The second condition in the first bullet is not a technicality for aircraft. A double integrator, $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, has $\text{Re}\{\lambda_i\} = 0$ yet $\underline{x}(t) = [x_1(0) + t x_2(0), x_2(0)]^T$ is unbounded. Strict negativity admits no such case, which is why placement targets $\text{Re}\{\lambda_i\} < 0$.

2. Controllability and Observability

Consider

$$\begin{cases} \dot{\underline{x}}(t) = \mathbf{A}\underline{x}(t) + \mathbf{B}\underline{u}(t) \\ \underline{y}(t) = \mathbf{C}\underline{x}(t) \end{cases}$$

The system is *controllable* if given any initial state $\underline{x}_0$ and any desired finite state $\underline{x}_1$, we can find a finite time t_1 and a bounded control $\underline{u}(t)$ ($t \in [0, t_1]$) to change from $\underline{x}(0) = \underline{x}_0$ to $\underline{x}(t_1) = \underline{x}_1$.

The system is *observable* if there exists a finite time $t_1 > 0$ such that $\underline{x}_0$ can be determined with a knowledge of $\underline{u}(t)$ and $\underline{y}(t)$ for $t \in [0, t_1]$, for any $\underline{u}$.

Controllability Matrix

$$\mathbf{S} = \begin{bmatrix} \mathbf{B} & \vdots & \mathbf{AB} & \vdots & \dots & \vdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$$

The system is controllable iff $\text{rank}(\mathbf{S}) = n$.

Observability Matrix

$$\mathbf{Q} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}$$

The system is observable iff $\text{rank}(\mathbf{Q}) = n$.

Controllability is what the rest of this lecture needs: it is the condition under which the poles can be placed at all. Observability is its dual and is not used here — it is the condition for *reconstructing* the state from the output, and it is where Lecture 10 begins.

[Example 2] Consider the following state-space equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ \epsilon & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} u$$

where all elements a_{11} , a_{12} , a_{22} , b are non-zero, except the parameter ϵ . The controllability matrix becomes

$$\mathbf{S} = [\mathbf{B} \mid \mathbf{AB}] = \left[\begin{array}{c|c} b & a_{11}b \\ 0 & \epsilon b \end{array} \right]$$

- ▶ controllable when $\epsilon \neq 0$
- ▶ uncontrollable if $\epsilon = 0$. When $\epsilon = 0$, the system is decoupled $\dot{x}_2 = a_{22}x_2$, that cannot be assigned by any control effort.



[Example 3] Mass-Spring-Damping System

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0 \cdot u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{\mathbf{x}} = x_1 - \text{always observable}$$

$$\underline{\mathbf{Q}} = \begin{bmatrix} \underline{\mathbf{c}} \\ \underline{\mathbf{cA}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \underline{\mathbf{x}} = x_2 - \text{observable when } k \neq 0$$

$$\underline{\mathbf{Q}} = \begin{bmatrix} \underline{\mathbf{c}} \\ \underline{\mathbf{cA}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}$$

3. Pole Placement: Standard Formula

$$\begin{aligned}\dot{\underline{x}} &= \underline{A}\underline{x} + \underline{b}u, \quad \underline{A} \in \mathcal{R}^{n \times n}, \underline{b} \in \mathcal{R}^n \\ u &= -\underline{k}^T \underline{x}\end{aligned}$$

where the control gain matrix $\underline{k}^T$ is a row vector

$\underline{k}^T = [k_1 \ k_2 \ \dots \ k_n]$ whose elements (gains) are to be determined.

- ▶ Assume the open-loop system is properly known by the model, its characteristic equation is given as:

$$a(s) = \det[s\mathbf{1} - \mathbf{A}] = s^n + a_1s^{n-1} + a_2s^{n-2} + \dots + a_{n-1}s + a_n$$

- ▶ Assume the closed-loop system has identified n desired poles (we will discuss this assumption shortly) that a desired characteristic equation is represented by:

$$\alpha(s) = s^n + \alpha_1s^{n-1} + \alpha_2s^{n-2} + \dots + \alpha_{n-1}s + \alpha_n$$

On the other hand, the closed-loop system by taking the state feedback $u = -\underline{k}^T \underline{x}$ becomes

$$\dot{\underline{x}} = (\underline{A} - \underline{b}\underline{k}^T)\underline{x}$$

with the characteristic equation

$$\det[s\mathbf{1} - (\underline{A} - \underline{b}\underline{k}^T)]$$

leading to

$$\alpha(s) = a(s)[1 + \underline{k}^T(s\mathbf{1} - \underline{A})^{-1}\underline{b}]$$

Leverrier's Theorem: $(s\mathbf{1} - \mathbf{A})^{-1}$

The inverse matrix of $(s\mathbf{1} - \mathbf{A})$ is represented by

$$(s\mathbf{1} - \mathbf{A})^{-1} = \frac{1}{\det[s\mathbf{1} - \mathbf{A}]}(s\mathbf{1} - \mathbf{A})^A$$

where $(s\mathbf{1} - \mathbf{A})^A$ is the companion matrix that is described as follows:

$$(s\mathbf{1} - \mathbf{A})^A = s^{n-1}\mathbf{F}_1 + \dots s\mathbf{F}_{n-1} + \mathbf{F}_n$$

then the inverse matrix is given by:

$$\begin{aligned}(s\mathbf{1} - \mathbf{A})^{-1} &= \frac{1}{\det[s\mathbf{1} - \mathbf{A}]}(s\mathbf{1} - \mathbf{A})^A \\ &= \frac{s^{n-1}\mathbf{F}_1 + \dots s\mathbf{F}_{n-1} + \mathbf{F}_n}{s^n + a_1s^{n-1} + a_2s^{n-2} + \dots + a_{n-1}s + a_n}\end{aligned}$$

and the coefficients are obtained recursively:

$$\mathbf{F}_1 = \mathbf{1}, a_1 = -\text{tr}(\mathbf{A})$$

$$\vdots$$

$$a_k = -\frac{1}{k} \text{tr}(\mathbf{A}\mathbf{F}_k), \quad k = 1, 2, \dots, n-1$$

$$\mathbf{F}_{k+1} = \mathbf{A}\mathbf{F}_k + a_k \mathbf{1}$$

$$\vdots$$

$$a_n = -\frac{1}{n} \text{tr}(\mathbf{A}\mathbf{F}_n) \quad \text{tr} = \text{trace}$$

Relationship

$$[\alpha_1 \ \alpha_2 \ \dots \ \alpha_n] = [a_1 \ a_2 \ \dots \ a_n] + \underline{k}^T [\underline{b} \ | \underline{A}\underline{b} \ | \ \dots \ | \underline{A}^{n-1}\underline{b}] \cdot \begin{bmatrix} 1 & a_1 & \dots & a_{n-1} \\ & 1 & \dots & a_{n-2} \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}$$

- The matrix $[\underline{b} \ | \underline{A}\underline{b} \ | \ \dots \ | \underline{A}^{n-1}\underline{b}]$ is called the controllability matrix, denoted by $\mathbf{S}$

$$\underline{\alpha}^T = \underline{a}^T + \underline{k}^T \mathbf{S} \mathbf{V}_{[n-1]}$$

leading to the solution for $\underline{k}^T$ by

$$\underline{k}^T = (\underline{\alpha}^T - \underline{a}^T) \mathbf{V}_{[n-1]}^{-1} \mathbf{S}^{-1}$$

[Example 4] B747-100 short period. From Lecture 03, retaining the fast states $\underline{x} = [w \ q]^T$ with the elevator as input,

$$\dot{\underline{x}} = \begin{bmatrix} -0.3128 & 235.9 \\ -0.003362 & -0.4282 \end{bmatrix} \underline{x} + \begin{bmatrix} -5.4715 \\ -1.1569 \end{bmatrix} \delta_e$$

with w in m/s, q in rad/s and δ_e in rad. Unlike the companion matrices of the previous examples, $\mathbf{A}$ is *not* in canonical form, so the coefficients must actually be computed:

$$a(s) = \det[s\mathbf{1} - \mathbf{A}] = s^2 + 0.7410s + 0.9272, \quad \underline{a}^T = [0.7410 \ 0.9272]$$

giving open-loop poles $-0.3705 \pm 0.8888i$, i.e. $\zeta = 0.385$, $\omega_n = 0.963$ rad/s — just inside the Level 1 box.

$$\mathbf{S} = [\underline{b} | \mathbf{A} \underline{b}] = \begin{bmatrix} -5.4715 & -271.2120 \\ -1.1569 & 0.5138 \end{bmatrix}, \quad \det \mathbf{S} = -316.59 \neq 0$$

so the pair is controllable, and

$$\mathbf{S}^{-1} = \begin{bmatrix} -0.001623 & -0.856670 \\ -0.003654 & 0.017283 \end{bmatrix}$$

Take the Lecture 04 handling-qualities target $\zeta = 0.7$, $\omega_n = 1.0$ rad/s, i.e. $p_{1,2} = -0.7 \pm 0.714i$:

$$\alpha(s) = s^2 + 1.4000s + 1.0000, \quad \underline{\alpha}^T - \underline{a}^T = [0.6590 \quad 0.0728]$$

Constructing $\mathbf{V}_{[n-1]}$ for $n = 2$,

$$\begin{aligned} \mathbf{V}_{[n-1]} &= \begin{bmatrix} 1 & a_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.7410 \\ 0 & 1 \end{bmatrix} \\ \mathbf{V}_{[n-1]}^{-1} &= \begin{bmatrix} 1 & -0.7410 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

$$\underline{k}^T = (\underline{\alpha}^T - \underline{a}^T) \mathbf{V}_{[n-1]}^{-1} \mathbf{S}^{-1} = [0.000449 \quad -0.571728]$$

Verifying, $\text{eig}(\mathbf{A} - \mathbf{b}\mathbf{k}^T) = -0.7 \pm 0.714i$ as required. A perturbation of $w = 10 \text{ m/s}$ calls for 0.26° of elevator, and $q = 5^\circ/\text{s}$ for -2.86° : the two gains differ by three orders of magnitude only because w is in m/s while q is in rad/s .



4. Pole Placement: Canonical Form

The previous section already produces a gain. This second derivation is worth following for a different reason: it builds the *controllable canonical form*, the realisation in which the characteristic coefficients appear directly as a row of **A**. That structure returns in the observer and estimator constructions of Lecture 10, so the transformation is reused even though the gain it yields here is the same one.

Consider the general case of strictly proper transfer function

$$\frac{y(s)}{u(s)} = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \dots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}$$

Canonical Form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$y = \begin{bmatrix} b_n & b_{n-1} & \dots & b_2 & b_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

in the format of

$$\begin{cases} \dot{\underline{\hat{x}}}(t) &= \hat{\mathbf{A}}\underline{\hat{x}}(t) + \underline{\varepsilon}_n u(t) \\ y(t) &= \underline{\hat{c}}^T \underline{\hat{x}}(t) \end{cases}$$

Consider the general ss format:

$$\begin{cases} \dot{\underline{x}}(t) &= \mathbf{A}\underline{x}(t) + \underline{b}u(t) \\ y(t) &= \underline{c}^T \underline{x}(t) \end{cases}$$

choose

$$\mathbf{T} = \mathbf{S}\mathbf{W}_{[n-1]} = \left[\begin{array}{c|c|c|c} \underline{\mathbf{b}} & \mathbf{A}\underline{\mathbf{b}} & \dots & \mathbf{A}^{n-1}\underline{\mathbf{b}} \end{array} \right] \cdot \begin{bmatrix} a_{n-1} & \dots & a_1 & 1 \\ \vdots & \dots & & \\ a_1 & & 1 & \\ 1 & & & \end{bmatrix}$$

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \hat{\mathbf{A}}, \quad \hat{\underline{\mathbf{b}}} = \mathbf{T}^{-1}\underline{\mathbf{b}} = \underline{\varepsilon}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \hat{\underline{\mathbf{c}}}^T = \underline{\mathbf{c}}^T \mathbf{T}$$

introduce a new state converting from $\underline{x}$ to $\underline{\hat{x}} = \mathbf{T}^{-1}\underline{x}$, then the state-space equation becomes

$$\begin{aligned}\dot{\underline{\hat{x}}} &= \mathbf{T}^{-1}\dot{\underline{x}} \\ &= \mathbf{T}^{-1}(\mathbf{A}\underline{x} + \underline{b}u) \\ &= \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\underline{\hat{x}} + \mathbf{T}^{-1}\underline{b}u \\ &= \hat{\mathbf{A}}\underline{\hat{x}} + \underline{\varepsilon}_n u\end{aligned}$$

where $\hat{\mathbf{A}}$ has the control canonical form, and

$$G(s) = \underline{c}^T (s\mathbf{1} - \mathbf{A})^{-1} \underline{b} = \underline{\hat{c}}^T (s\mathbf{1} - \hat{\mathbf{A}})^{-1} \underline{\hat{b}}$$

Following the state conversation, the control law becomes

$$u = -\underline{\mathbf{k}}^T \underline{\mathbf{x}} = -\underline{\mathbf{k}}^T \mathbf{T} \hat{\underline{\mathbf{x}}} = -\hat{\underline{\mathbf{k}}}^T \hat{\underline{\mathbf{x}}}$$

Let $\hat{\underline{\mathbf{k}}}^T = \underline{\mathbf{k}}^T \mathbf{T} = [\delta_n \ \dots \ \delta_1]$.

$$\begin{aligned} \det(s\mathbf{1} - \mathbf{A} + \underline{\mathbf{b}}\underline{\mathbf{k}}^T) &= \det(s\mathbf{1} - \hat{\mathbf{A}} + \hat{\underline{\mathbf{b}}}\hat{\underline{\mathbf{k}}}^T) \\ &= \det(s\mathbf{1} - \hat{\mathbf{A}} + \underline{\varepsilon}_n \hat{\underline{\mathbf{k}}}^T) \end{aligned}$$

leading to

$$\hat{\underline{\mathbf{k}}}^T = \underline{\mathbf{k}}^T \mathbf{T} = [\delta_n \ \dots \ \delta_1] = [\alpha_n - a_n \ \dots \ \alpha_1 - a_1]$$

If we give a matrix expression $\underline{\alpha}_r^T = [\alpha_n \ \dots \ \alpha_1]$, $\underline{a}_r^T = [a_n \ \dots \ a_1]$, then we obtain the alternative control law design formula as follows:

$$\underline{\mathbf{k}}^T = \hat{\underline{\mathbf{k}}}^T \mathbf{T}^{-1} = [\underline{\alpha}_r^T - \underline{a}_r^T] \mathbf{W}_{[n-1]}^{-1} \mathbf{S}^{-1}$$

[Example 5] Same B747-100 short period as before, with the coefficient vectors written in reverse order:

$$\underline{\mathbf{a}}_r^T = [a_2 \ a_1] = [0.9272 \ 0.7410], \quad \underline{\boldsymbol{\alpha}}_r^T = [1.0000 \ 1.4000]$$

For $n = 2$,

$$\mathbf{W}_{[n-1]} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0.7410 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{W}_{[n-1]}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -0.7410 \end{bmatrix}$$

$$\underline{\mathbf{k}}^T = (\underline{\boldsymbol{\alpha}}_r^T - \underline{\mathbf{a}}_r^T) \mathbf{W}_{[n-1]}^{-1} \mathbf{S}^{-1} = [0.000449 \ -0.571728]$$

the same gain as Example 4, by a different route.



5. Pole Placement: Ackermann's Formula

The so-called Ackermann's formula provides a third design approach for pole placement. Consider the system Eq. (20), the closed-loop state matrix after state feedback is given by

$$\mathbf{A}_c = \mathbf{A} - \underline{\underline{b}}\underline{\underline{k}}^T$$

Cayley-Hamilton Theorem

Recall the characteristic equation

$$\phi(s) = \det(s\mathbf{I} - \mathbf{A}) = s^n + a_1s^{n-1} + \dots + a_{n-1}s + a_n$$

Then the Cayley-Hamilton Theorem states that

$$\phi(\mathbf{A}) = \mathbf{A}^n + a_1\mathbf{A}^{n-1} + \dots + a_{n-1}\mathbf{A} + a_n\mathbf{I} = \mathbf{0}$$

In the case $n = 3$,

$$\phi(\mathbf{A}_c) = \mathbf{A}_c^3 + \alpha_1 \mathbf{A}_c^2 + \alpha_2 \mathbf{A}_c + \alpha_3 \mathbf{1} = \mathbf{0}$$

substituted by the relationship associated with $\mathbf{A}_c = \mathbf{A} - \underline{\mathbf{b}}\underline{\mathbf{k}}^T$, we have

$$\phi(\mathbf{A}) = \begin{bmatrix} \underline{\mathbf{b}} & : & \mathbf{A}\underline{\mathbf{b}} & : & \mathbf{A}^2\underline{\mathbf{b}} \end{bmatrix} \begin{bmatrix} \alpha_2 \underline{\mathbf{k}}^T + \alpha_1 \underline{\mathbf{k}}^T \mathbf{A}_c + \underline{\mathbf{k}}^T \mathbf{A}_c^2 \\ \alpha_1 \underline{\mathbf{k}}^T + \underline{\mathbf{k}}^T \mathbf{A}_c \\ \underline{\mathbf{k}}^T \end{bmatrix}$$

In other words, we have obtained the relationship

$$\phi(\mathbf{A}) = \mathbf{S} \begin{bmatrix} \alpha_2 \underline{\mathbf{k}}^T + \alpha_1 \underline{\mathbf{k}}^T \mathbf{A}_c + \underline{\mathbf{k}}^T \mathbf{A}_c^2 \\ \alpha_1 \underline{\mathbf{k}}^T + \underline{\mathbf{k}}^T \mathbf{A}_c \\ \underline{\mathbf{k}}^T \end{bmatrix}$$

where $\mathbf{S}$ is the controllability matrix and the control law to be determined $\underline{\mathbf{k}}^T$ happens to be at the last block, i.e.,

$$\underline{\mathbf{k}}^T = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \alpha_2 \underline{\mathbf{k}}^T + \alpha_1 \underline{\mathbf{k}}^T \mathbf{A}_c + \underline{\mathbf{k}}^T \mathbf{A}_c^2 \\ \alpha_1 \underline{\mathbf{k}}^T + \underline{\mathbf{k}}^T \mathbf{A}_c \\ \underline{\mathbf{k}}^T \end{bmatrix}$$

When we are dealing with the scalar input u , and the system is controllable, i.e., $\mathbf{S}^{-1}$ exists, then we can calculate $\underline{\mathbf{k}}^T$ as

$$\underline{\mathbf{k}}^T = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{b}} & : & \mathbf{A}\underline{\mathbf{b}} & : & \mathbf{A}^2\underline{\mathbf{b}} \end{bmatrix}^{-1} \phi(\mathbf{A})$$

Ackermann's Theorem

This conclusion can be extended to an arbitrary order of n , as given by the so-called Ackermann's formula

$$\underline{k}^T = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix} \mathbf{S}^{-1} \phi(\mathbf{A}) = \underline{\epsilon}_n^T \mathbf{S}^{-1} \phi(\mathbf{A})$$

Read this as an *existence proof*: for any controllable pair $(\mathbf{A}, \underline{b})$ a gain placing any chosen pole set always exists, and here it is in closed form. It is not the numerical method — forming $\phi(\mathbf{A})$ and inverting $\mathbf{S}$ are both ill-conditioned. MATLAB's `acker` implements exactly this formula and degrades rapidly beyond $n \simeq 10$.

[Example 6] Same B747-100 short period. With

$$\phi(s) = s^2 + 1.4s + 1.0,$$

$$\phi(\mathbf{A}) = \mathbf{A}^2 + 1.4\mathbf{A} + 1.0 \cdot \mathbf{1} = \begin{bmatrix} -0.1334 & 155.46 \\ -0.002216 & -0.2094 \end{bmatrix}$$

and with $\mathbf{S}^{-1}$ as above,

$$\underline{k}^T = \underline{\varepsilon}_2^T \mathbf{S}^{-1} \phi(\mathbf{A}) = [0.000449 \quad -0.571728]$$

the same gain again — as it must be. For a single input $\dim \mathcal{S}_\lambda = 1$, so the eigenvector directions are forced and the gain is unique; the three formulas are three routes to one answer.



What the Gain Buys

A gust leaves $\underline{x}_0 = [10 \ 0]^T$ — 10 m/s of downwash, no pitch rate — and the elevator closes the loop through $\delta_e = -\underline{k}^T \underline{x}$, using the gain of Examples 4–6.

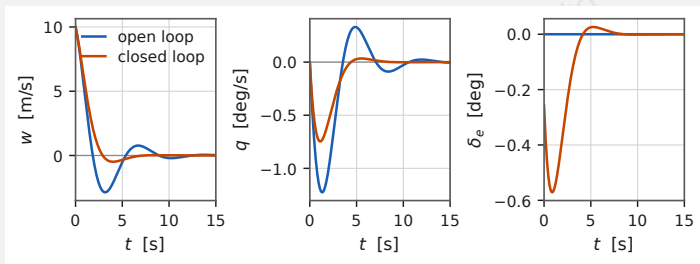


Figure 1: B747-100 short period: open loop at $\zeta = 0.385$, closed loop at $\zeta = 0.7$.

Settling time halves ($10.5 \rightarrow 5.7$ s) and peak pitch rate drops from 1.22 to $0.75^\circ/\text{s}$, for 0.57° of elevator — a number nothing in the design asked for.

Summary of Pole Placement Design Methods

- (I) the standard pole placement: $\underline{k}^T = (\underline{\alpha}^T - \underline{a}^T) \mathbf{V}_{[n-1]}^{-1} \mathbf{S}^{-1}$
- (II) the canonical form pole placement: $\underline{k}^T = (\underline{\alpha}_r^T - \underline{a}_r^T) \mathbf{W}_{[n-1]}^{-1} \mathbf{S}^{-1}$
- (III) the Ackermann's formula: $\underline{k}^T = \underline{\varepsilon}_n^T \mathbf{S}^{-1} \phi(\mathbf{A})$

The Full Model, in MATLAB

The three examples used the 2×2 short period so the algebra stays visible. The full longitudinal model is $n = 4$ and nobody inverts $\mathbf{S} = [\underline{\mathbf{b}} \ \mathbf{A}\underline{\mathbf{b}} \ \mathbf{A}^2\underline{\mathbf{b}} \ \mathbf{A}^3\underline{\mathbf{b}}]$ by hand: its entries span 10^{-4} to 271 and $\kappa(\mathbf{S}) = 2.9 \times 10^3$.

```
p = [-0.7+0.714i, -0.7-0.714i, -0.05+0.05i, -0.05-0.05i];
K = place(A,B,p);           % or acker(A,b,p) for a single input
eig(A-B*K)                  % ALWAYS verify
```

This places all four poles — short period at $\zeta = 0.7$, $\omega_n = 1.0$, and the phugoid moved out to $-0.05 \pm 0.05i$ — giving

$$\underline{\mathbf{k}}^T = [0.00031 \quad 0.0012 \quad -0.65404 \quad -0.26854]$$

The formulas still hold at $n = 4$: computed by hand the standard formula and Ackermann agree to 9×10^{-15} and the poles land within 3×10^{-14} of their targets. What changes is only who does the arithmetic. Note also that placing four poles means *choosing* where the phugoid goes — a question the formulas do not answer.

Why Ackermann Is Not Enough

1. It is numerically fragile. The formula inverts the controllability matrix $\mathbf{S} = [\underline{\mathbf{b}} \ \mathbf{A}\underline{\mathbf{b}} \ \dots \ \mathbf{A}^{n-1}\underline{\mathbf{b}}]$ and forms $\phi(\mathbf{A})$ from high powers of $\mathbf{A}$.

The columns of $\mathbf{S}$ become nearly parallel as $\mathbf{A}^k \underline{\mathbf{b}}$ aligns with the dominant eigenvector:

n	3	7	11	13
$\kappa(\mathbf{S})$	4×10^1	1×10^8	3×10^{16}	1×10^{21}
pole error	2×10^{-15}	1×10^{-11}	1×10^{-8}	2×10^{-7}

By $n = 11$, $\kappa(\mathbf{S})$ exceeds $1/\epsilon_{\text{mach}}$: $\mathbf{S}$ is numerically singular and the computed gain is meaningless.

2. It answers the wrong question when $m > 1$. Built for a scalar input, it does not extend — yet multi-input is exactly where a *choice* of gain exists, and valid choices differ by a factor of 100 in sensitivity.

We need a formulation that avoids inverting $\mathbf{S}$, admits $m > 1$, and makes the leftover freedom visible.

Start from the Eigenvector, Not the Polynomial

All three formulas match coefficients of a characteristic polynomial. The route that meets those three requirements starts somewhere else — from the eigenvectors.

If λ is to be a closed-loop eigenvalue with eigenvector $\underline{\mathbf{v}}$, then

$(\mathbf{A} - \mathbf{BK})\underline{\mathbf{v}} = \lambda \underline{\mathbf{v}}$, that is

$$(\mathbf{A} - \lambda \mathbf{1})\underline{\mathbf{v}} = \mathbf{B}(\mathbf{K}\underline{\mathbf{v}}).$$

The right-hand side is *some* vector in the range of $\mathbf{B}$, whatever $\mathbf{K}$ turns out to be. So $\underline{\mathbf{v}}$ is not free: it must lie in

$$\mathcal{S}_\lambda = \left\{ \underline{\mathbf{v}} : (\mathbf{A} - \lambda \mathbf{1})\underline{\mathbf{v}} \in \text{range } \mathbf{B} \right\},$$

obtained as the null space of the $n \times (n + m)$ matrix $[\mathbf{A} - \lambda \mathbf{1} \quad -\mathbf{B}]$.

One Fact Explains Everything

$$\dim \mathcal{S}_\lambda = m$$

the number of independent inputs.

Single input, $m = 1$. $\mathcal{S}_\lambda$ is a *line*. The eigenvector direction is forced, every $\underline{v}_j$ is determined by its λ_j , and so is $\underline{k}$. This is why the three formulas of the previous section must all return the same gain — there is only one.

Several inputs, $m > 1$. $\mathcal{S}_\lambda$ is m -dimensional. For each eigenvalue you may *choose* the eigenvector within an m -dimensional subspace. That choice is precisely the leftover freedom counted earlier: mn gain entries against n pole conditions.

Placement fixes the *eigenvalues*. The *eigenvectors* remain yours to pick, and they are what is left to design with.

What the Eigenvectors Are Worth: Bauer–Fike

Collect the chosen eigenvectors into $\mathbf{E} = [\underline{\mathbf{v}}_1 \ \dots \ \underline{\mathbf{v}}_n]$, so that

$$\mathbf{A}_c = \mathbf{E} \mathbf{\Lambda} \mathbf{E}^{-1}.$$

Bauer–Fike theorem. If $\mathbf{A}_c$ is perturbed to $\mathbf{A}_c + \mathbf{E}$, every eigenvalue μ of the perturbed matrix satisfies

$$\min_j |\mu - \lambda_j| \leq \kappa(\mathbf{E}) \|\mathbf{E}\|, \quad \kappa(\mathbf{E}) = \|\mathbf{E}\| \|\mathbf{E}^{-1}\|.$$

The poles you placed stay where you put them only to within $\kappa(\mathbf{E})\|\mathbf{E}\|$. Modelling error, gain rounding and actuator tolerance all enter as $\mathbf{E}$, so a design with a large $\kappa(\mathbf{E})$ has poles that are nominal only.

$\kappa(\mathbf{E}) \geq 1$ always, with equality when the eigenvectors are orthogonal.

The Kautsky–Nichols–Van Dooren Criterion

This turns the leftover freedom into a well-posed problem:

choose $\underline{\mathbf{v}}_j \in \mathcal{S}_{\lambda_j}$, for $j = 1, \dots, n$, so that $\mathbf{E}$ is as well conditioned as possible.

Minimising $\kappa(\mathbf{E})$ makes the assigned poles as insensitive as the system allows, and as a by-product usually keeps $\|\mathbf{K}\|$ small.

Once the $\underline{\mathbf{v}}_j$ are fixed, the gain follows from $\mathbf{A} - \mathbf{BK} = \mathbf{E}\mathbf{\Lambda}\mathbf{E}^{-1}$:

$$\mathbf{BK} = \mathbf{A} - \mathbf{E}\mathbf{\Lambda}\mathbf{E}^{-1},$$

solved in the least-squares sense when $\mathbf{B}$ is not square.

The algorithm that performs the minimisation — successive QR factorisations projecting each $\underline{\mathbf{v}}_j$ onto its admissible subspace — is the content of Kautsky, Nichols & Van Dooren, *Int. J. Control* **41** (1985) 1129–1155, and is beyond our scope. The *criterion* is the part worth carrying away.

Short-Period Example: Same Poles, Three Gains

The two-input short-period model, all three designs placing $\lambda = -5.6 \pm 4.2i$:

design	$\max K_{ij} $	$\kappa(\mathbf{E})$
free parameter $n = 0$	16.1	—
free parameter $n = 10$	1788.9	102.0
place (KNV)	16.1	1.0

Perturbing every entry of $\mathbf{K}$ by 0.01%:

design	resulting poles
$n = 10$	$-5.52 \pm 5.77i$
place (KNV)	$-5.598 \pm 4.199i$

Identical nominal poles; one design holds them, the other does not.
 $\kappa(\mathbf{E}) = 1$ means orthogonal eigenvectors — the best this plant allows.

place and acker

```
K = place(A,B,p)    % Kautsky-Nichols-Van Dooren: MIMO, robust  
k = acker(A,b,p)    % Ackermann's formula: SISO only, fragile
```

- ▶ `acker` is the closed form of the previous section. Use it to confirm a hand calculation on a small problem, not to design.
- ▶ `place` solves the eigenstructure problem above. For a single input it returns the same unique gain, because $\dim \mathcal{S}_\lambda = 1$ leaves nothing to choose.
- ▶ `place` refuses to assign a pole with multiplicity greater than rank **B**: there are not enough independent eigenvector directions. `acker` will accept it, which is one of the few cases where it is the only option.

Both place the poles. Only one of them chooses *which* of the infinitely many gains that place those poles you actually get.

What Placement Leaves Out: the Cost of Control

Pole placement asks one question — where should the closed-loop poles be? — and answers it exactly. It never asks what the control costs.

Same model, same method, only the poles pushed further left:

desired poles	$\max K_{ij} $	peak $ u $ from $\underline{x}_0 = [0.2 \ 0]^T$	settling
$-2.0 \pm 1.5i$	2.5	0.50 rad	1.94 s
$-5.6 \pm 4.2i$	16.1	3.21 rad	0.69 s
$-11.2 \pm 8.4i$	39.6	7.91 rad	0.35 s
$-22.4 \pm 16.8i$	86.6	17.3 rad	0.17 s

Every row meets its pole specification *exactly*, yet neither $\underline{k}^T = \underline{\varepsilon}_n^T \mathbf{S}^{-1} \phi(\mathbf{A})$ nor `place(A,B,p)` contains $\underline{u}$ anywhere. The actuator demand grows with no warning from the method, and by the second row already exceeds what an elevator can deliver.

Faster poles are not free, and placement never prices them. What follows replaces the guess with a cost that prices state error and control effort together, and returns a single gain.

Finding Optimal Solution

Linear Quadratic Performance Index:

$$J = \frac{1}{2} \int_0^{\infty} \left(\underline{x}^T \mathbf{Q} \underline{x} + \underline{u}^T \mathbf{R} \underline{u} \right) dt$$

where $\mathbf{Q} = \mathbf{Q}^T \geq 0$, $\mathbf{R} = \mathbf{R}^T > 0$.

The notation of matrix $\mathbf{R} = \mathbf{R}^T > 0$ means that the matrix $\mathbf{R}$ is symmetric, and is *positive definite*.

to minimize (optimize) an objective function (quadratic performance index) subject to the trajectory of the variables constrained by its dynamics function:

$$\min_{\underline{\mathbf{u}}} J = \frac{1}{2} \int_0^{\infty} \left(\underline{\mathbf{x}}^T \mathbf{Q} \underline{\mathbf{x}} + \underline{\mathbf{u}}^T \mathbf{R} \underline{\mathbf{u}} \right) dt = \int_0^{\infty} l \left(\underline{\mathbf{x}}(t), \underline{\mathbf{u}}(t) \right) dt$$

$$\text{subject to } \dot{\underline{\mathbf{x}}} = \mathbf{A} \underline{\mathbf{x}} + \mathbf{B} \underline{\mathbf{u}}, \text{ or } \dot{\underline{\mathbf{x}}} = \underline{\mathbf{f}} \left(\underline{\mathbf{x}}(t), \underline{\mathbf{u}}(t) \right)$$

Why Quadratic Format

Linear Quadratic Performance Index:

$$J = \frac{1}{2} \int_0^{\infty} \left(\underline{x}^T \mathbf{Q} \underline{x} + \underline{u}^T \mathbf{R} \underline{u} \right) dt$$

- ▶ energy of states $\underline{x}$
- ▶ energy of inputs $\underline{u}$

augmented optimization function

$$\min \hat{J} = \int_0^{\infty} [I + \underline{\lambda}^T (\underline{f} - \underline{\dot{x}})] dt \quad s.t. \quad \underline{\dot{x}} - \underline{f} = 0$$

where the Lagrange multiplier $\underline{\lambda}$ is also called the co-state vector.

define the Hamiltonian function

$$\mathcal{H} = l + \underline{\lambda}^T \underline{f}$$

the augmented optimization problem is now represented by

$$\min \hat{J} = \int_0^\infty [\mathcal{H} - \underline{\lambda}^T \underline{\dot{x}}] dt \quad s.t. \quad \underline{\dot{x}} - \underline{f} = 0$$

To solve this augmented optimization problem, we can apply the method of the first-order variation here. Let $\delta \underline{\mathbf{x}}$, $\delta \underline{\mathbf{u}}$ denote deviations from the optimum trajectory and control, then

$$\begin{aligned}\delta \hat{J} &= \int_0^\infty \left[\underline{\mathbf{H}}_{\underline{\mathbf{x}}}^T \delta \underline{\mathbf{x}} + \underline{\mathbf{H}}_{\underline{\mathbf{u}}}^T \delta \underline{\mathbf{u}} - \underline{\boldsymbol{\lambda}}^T \delta \dot{\underline{\mathbf{x}}} \right] dt \\ &= \int_0^\infty \left[\underline{\mathbf{H}}_{\underline{\mathbf{x}}}^T \delta \underline{\mathbf{x}} + \underline{\mathbf{H}}_{\underline{\mathbf{u}}}^T \delta \underline{\mathbf{u}} + \dot{\underline{\boldsymbol{\lambda}}}^T \delta \underline{\mathbf{x}} \right] dt - \underline{\boldsymbol{\lambda}}^T \delta \underline{\mathbf{x}} \Big|_0^\infty\end{aligned}$$

where $\underline{\mathbf{H}}_{\underline{\mathbf{x}}} = \frac{\partial \mathcal{H}}{\partial \underline{\mathbf{x}}}$, $\underline{\mathbf{H}}_{\underline{\mathbf{u}}} = \frac{\partial \mathcal{H}}{\partial \underline{\mathbf{u}}}$. For a given non-zero state $\underline{\mathbf{x}}_0$ and expecting zero steady-state $\underline{\mathbf{x}}_\infty = 0$, the last term of variations is zero.

Therefore, the constrained optimization problem requires the following first-order variations become zero:

$$\underline{H}_u = \frac{\partial \mathcal{H}}{\partial \underline{u}} = 0$$

$$\dot{\underline{\lambda}} = -\underline{H}_x = -\frac{\partial \mathcal{H}}{\partial \underline{x}} \quad \text{with naturally}$$

$$\dot{\underline{x}} = \underline{f} = \frac{\partial \mathcal{H}}{\partial \underline{\lambda}} \quad \text{since } \mathcal{H} = l + \underline{\lambda}^T \underline{f}$$

This is called the *Pontryagin Minimum Principle* [**Anderson2007**].

First of all, the Hamiltonian for the LTI system with the linear quadratic performance index becomes:

$$\mathcal{H} = \underline{\lambda}^T \underline{f} + l = \underline{\lambda}^T (\underline{A}\underline{x} + \underline{B}\underline{u}) + \frac{1}{2}(\underline{x}^T \underline{Q}\underline{x} + \underline{u}^T \underline{R}\underline{u})$$

According to the minimum principle,

$$\frac{\partial \mathcal{H}}{\partial \underline{u}} = \underline{B}^T \underline{\lambda} + \underline{R}\underline{u} = 0$$

$$\underline{u}^* = -\underline{R}^{-1} \underline{B}^T \underline{\lambda}$$

The corresponding Hamiltonian becomes

$$\begin{aligned}\mathcal{H}^* &= \underline{\lambda}^T \underline{f} + l = \underline{\lambda}^T (\underline{A}\underline{x} + \underline{B}\underline{u}) + \frac{1}{2}(\underline{x}^T \underline{Q}\underline{x} + \underline{u}^T \underline{R}\underline{u}) \\ &= \underline{\lambda}^T \left(\underline{A}\underline{x} - \underline{B}\underline{R}^{-1}\underline{B}^T \underline{\lambda} \right) + \frac{1}{2} \left(\underline{x}^T \underline{Q}\underline{x} + (\underline{R}^{-1}\underline{B}^T \underline{\lambda})^T \underline{R} (\underline{R}^{-1}\underline{B}^T \underline{\lambda}) \right) \\ &= \underline{\lambda}^T \underline{A}\underline{x} + \frac{1}{2} \underline{x}^T \underline{Q}\underline{x} - \frac{1}{2} \underline{\lambda}^T \underline{B}\underline{R}^{-1}\underline{B}^T \underline{\lambda}\end{aligned}$$

Now, if we substitute the Hamiltonian of the minimum case $\mathcal{H}^*$ into the other two minimum principle formula,

$$\dot{\underline{x}} = \frac{\partial \mathcal{H}^*}{\partial \underline{\lambda}} = \underline{A}\underline{x} + \underline{B}\underline{u}^* = \underline{A}\underline{x} - \underline{R}^{-1}\underline{B}^T\underline{\lambda}$$

$$\dot{\underline{\lambda}} = -\frac{\partial \mathcal{H}^*}{\partial \underline{x}} = -\underline{Q}\underline{x} - \underline{A}^T\underline{\lambda}$$

leading to the following differential equation in a matrix format

$$\begin{bmatrix} \dot{\underline{x}} \\ \dot{\underline{\lambda}} \end{bmatrix} = \begin{bmatrix} \underline{A} & -\underline{B}\underline{R}^{-1}\underline{B}^T \\ -\underline{Q} & -\underline{A}^T \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{\lambda} \end{bmatrix}$$

In other words, the minimum principle gives an open-loop solution of $\underline{u}^* = -\mathbf{R}^{-1}\mathbf{B}^T\lambda$ that follows the trajectory given by the above equation. If the solution is denoted by $\underline{x}^*, \lambda^*$, then $\underline{u}^*$ is fully obtained. We are now ready to use this development to seek the state-feedback control solution.

Assume the solution is denoted by $\underline{x}^*(t)$, $\underline{\lambda}^*(t)$, and $\underline{\lambda}^*(t)$ has the property of $\underline{\lambda}^*(t) = \mathbf{P}\underline{x}^*(t)$, $\mathbf{P} = \mathbf{P}^T$, then

$$\dot{\underline{\lambda}}^* = -\mathbf{Q}\underline{x}^* - \mathbf{A}^T \underline{\lambda}^* = -(\mathbf{Q} + \mathbf{A}^T \mathbf{P})\underline{x}^*$$

on the other hand, based on the property $\underline{\lambda}^* = \mathbf{P}\underline{x}^*$,

$$\begin{aligned}\dot{\underline{\lambda}}^* &= \dot{\mathbf{P}}\underline{x}^* + \mathbf{P}\dot{\underline{x}}^* \\ &= \dot{\mathbf{P}}\underline{x}^* + \mathbf{P}[\mathbf{A}\underline{x} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\lambda^*] \\ &= [\dot{\mathbf{P}} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}]\underline{x}^*\end{aligned}$$

Therefore, the hypothetical relationship $\underline{\lambda}^*(t) = \mathbf{P}\underline{x}^*(t)$ exists if we can solve the following so-called *Riccati equation*

$$-\dot{\mathbf{P}} = \mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q} \quad \mathbf{P} = \mathbf{P}^T$$

When the system is completely controllable, there further exists a constant $\mathbf{P}$ that

$$\mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q} = 0 \quad \mathbf{P} = \mathbf{P}^T$$

which is called the *algebraic Riccati equation*. In addition, the solution $\mathbf{P}$ of the Riccati equation (or algebraic Riccati equation) gives the control $\underline{\mathbf{u}}^*$ the feedback law:

$$\underline{\mathbf{u}}^* = -\mathbf{R}^{-1}\mathbf{B}^T\lambda^* = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\underline{\mathbf{x}}^*, \text{ or } \underline{\mathbf{u}}^* = -\mathbf{K}\underline{\mathbf{x}}^*$$

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}$$

the calculated state-feedback gain $\mathbf{K}$ as shown is called the *Kalman gain*.

The closed-loop system:

$$\mathbf{A}_c = \mathbf{A} - \mathbf{B}\mathbf{K}$$

Assume there exists $\mathbf{P} = \mathbf{P}^T \geq 0$ s.t.

$$\frac{d}{dt}(\underline{x}^T \mathbf{P} \underline{x}) = -\underline{x}^T (\mathbf{Q} + \mathbf{K}^T \mathbf{R} \mathbf{K}) \underline{x}$$

on the other hand,

$$\begin{aligned} \frac{d}{dt}(\underline{x}^T \mathbf{P} \underline{x}) &= \dot{\underline{x}}^T \mathbf{P} \underline{x} + \underline{x}^T \mathbf{P} \dot{\underline{x}} \\ &= \underline{x}^T (\mathbf{A}_c^T \mathbf{P} + \mathbf{P} \mathbf{A}_c) \underline{x} \\ &= -\underline{x}^T (\mathbf{Q} + \mathbf{K}^T \mathbf{R} \mathbf{K}) \underline{x} \end{aligned}$$

substituting $\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}$ into the Lyapunov equation above gives the algebraic Riccati equation

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q} = 0$$

and the Kalman gain

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}$$

If $(\mathbf{A}, \mathbf{B})$ is stabilizable and $(\mathbf{A}, \mathbf{Q}^{1/2})$ is detectable, then

- ▶ the Riccati equation has a unique solution $\mathbf{P} = \mathbf{P}^T \geq 0$;
- ▶ the closed loop under $\underline{u} = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\underline{x}$ is asymptotically stable.

Both hypotheses are needed. Stabilizability alone leaves an unstable mode that $\mathbf{Q}$ cannot see, and no choice of weights will penalise it. If $(\mathbf{A}, \mathbf{Q}^{1/2})$ is observable rather than merely detectable, then $\mathbf{P} > 0$.

The PI

$$\begin{aligned} J &= \frac{1}{2} \int_0^{\infty} \left(\underline{x}^T \mathbf{Q} \underline{x} + \underline{u}^T \mathbf{R} \underline{u} \right) dt \\ &= -\frac{1}{2} \int_0^{\infty} \frac{d}{dt} (\underline{x}^T \mathbf{P} \underline{x}) dt \\ &= \frac{1}{2} \underline{x}_0^T \mathbf{P} \underline{x}_0 - \frac{1}{2} \lim_{t \rightarrow \infty} \underline{x}^T \mathbf{P} \underline{x} \end{aligned}$$

when the closed-loop system is asymptotically stable, $\underline{x} \rightarrow 0$,

$$J_{\min} = J^* = \frac{1}{2} \underline{x}_0^T \mathbf{P} \underline{x}_0$$

Summary

- 1 The State-Space Description
- 2 Controllability and Observability
- 3 Pole Placement: Standard Formula
- 4 Pole Placement: Canonical Form
- 5 Pole Placement: Ackermann's Formula
- 6 Robust Placement: the Eigenstructure View
- 7 From Placement to Optimal Control