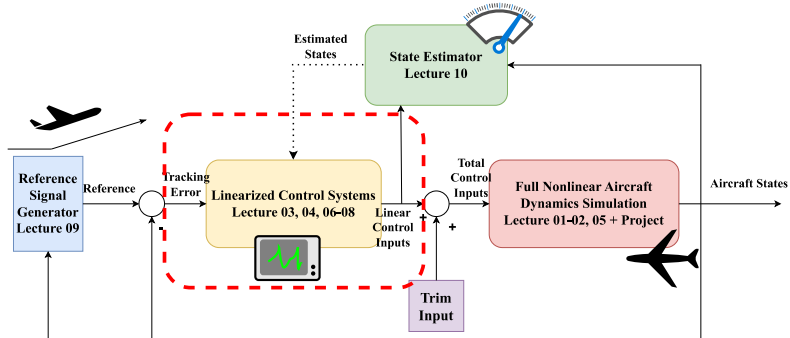


Lecture 04

Classical Feedback Flight Control

October 7, 2026

Acknowledgement: These slides were developed with reference to course materials by Professor Hugh H. T. Liu.



Contents

① Pitch Attitude Control

② Speed Control

Before We Start: Dynamic Response Review

- ▶ Classical feedback control design (this lecture) builds directly on dynamic response concepts – transfer functions, time and frequency response, stability margins – mostly covered in AER372.
- ▶ A self-study review of this material is provided in **R3** (Basic Control Systems Review); no lecture time is allocated to it.
- ▶ Please read R3 on your own *before* this lecture if you need a refresher – the design methods that follow assume you're comfortable with it.

Review: B747-100 Elevator-to-Pitch Channel

The plant for the design that follows is the elevator-to-pitch channel identified in Lecture 03. Its poles and zeros are

$$p_{1,2} = -0.3717 \pm 0.8869i \quad \text{short period}$$

$$p_{3,4} = -0.0033 \pm 0.0672i \quad \text{phugoid}$$

$$z_1 = -0.2944$$

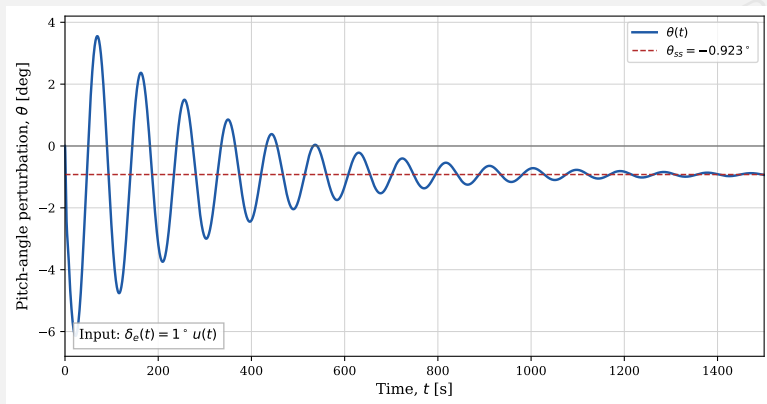
$$z_2 = -0.0113$$

All poles lie in the left half-plane, so the open-loop aircraft is stable. The lightly damped phugoid pair sits close to the imaginary axis and governs the long settling time. The DC gain is

$$G_{\theta\delta_e}(0) = \frac{-0.003864}{0.004187} = -0.923,$$

so a positive 1° elevator step gives $\theta_{ss} = -0.923^\circ$ under the course sign convention.

Review: Pitch Response to a 1° Elevator Step



The fast short-period transient decays first; the lightly damped phugoid then dominates the long-time response. The closed-loop designs in this lecture are judged against this baseline.

Review: Why the Open-Loop Response Is Not Enough

The open-loop response is stable, but its control performance is poor:

- ▶ the lightly damped phugoid produces a large, slowly decaying oscillation;
- ▶ the pitch angle ranges from approximately -6.14° to $+3.55^\circ$, a peak-to-peak excursion of 9.70° ;
- ▶ the 2% settling time is approximately 1710 s (28.5 min); and
- ▶ the DC gain is negative, giving $\theta_{ss} = -0.923^\circ$ for a positive 1° elevator step.

Feedback control must therefore raise the modal damping, cut the transient excursion and settling time, and deliver the required steady-state response without exceeding actuator limits. Those four requirements drive every design in this lecture.

1. Pitch Attitude Control

The characteristic lightly damped, low-frequency oscillation in pitch attitude was identified before, leads to large peaks and long transients. Pitch attitude control is a typical feedback control channel, that the pitch angle is regulated through the control of elevator angle. In this section, we will explore classical design approaches of finding proper controller $C(s)$ as illustrated in Fig.1. Again, we will continue using the [B747-100] case study to illustrate the design concepts and procedure.

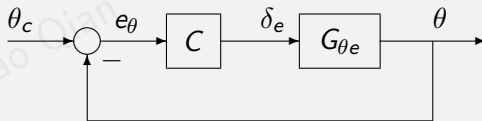


Figure 1: Pitch Attitude Control

First of all, the transfer functions

$$G_{\theta e} = \frac{-1.157s^2 - 0.3537s - 0.003864}{s^4 + 0.7499s^3 + 0.9341s^2 + 0.009449s + 0.004187}$$

where the open-loop zeros and poles are:

$$z_1 = -0.2944, z_2 = -0.0113,$$

$p_{1,2} = -0.0033 \pm 0.0672i, p_{3,4} = -0.3717 \pm 0.8869i$. In the meantime, its phugoid mode approximation

$$G_{\theta e}^{ph} = \frac{-1.412s^2 - 0.428s - 0.004675}{s^2 + 0.009685s + 0.005065}$$

also generating the open-loop zeros and poles:

$$z_1 = -0.2919, z_2 = -0.0113, p_{1,2} = -0.0048 \pm 0.0710i.$$

The following observations are made:

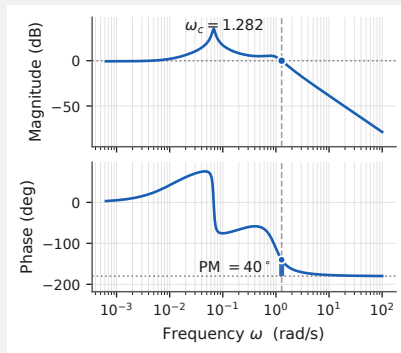
- ▶ The step response reveals the long settling time, it implies that the phugoid mode dominates the dynamic behaviour;
- ▶ the transfer function has a negative sign, indicating the physics of positive deflection of elevator angle driving negative pitch angle reaction (nose down); it is not a standard transfer function from control design perspective, which is often assumed to be 'positive'. One may use $-G(s)$ for the sake of carrying out design/analysis, to be reminded that a corresponding negative sign of control law will be added to compensate for the actual negative nature of the transfer function.
- ▶ Numerically, the steady-state value of θ responding to 1 degree of elevator deflection is settled to be close to -1 degree, $\theta_{ss} = \lim_{t \rightarrow \infty} \theta(t) = \lim_{s \rightarrow 0} s\theta(s) = -0.9$ (deg). The phugoid approximation gives:

$$\theta_{ss} = \frac{c_0}{C} \approx \frac{M_{\delta_e}}{M_w} \frac{X_u}{Z_u} \frac{Z_w}{mg} = -0.82 \text{ (deg)}.$$

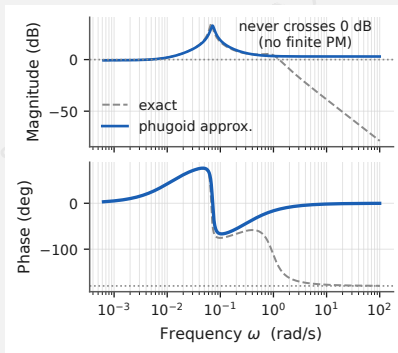
To summarize, the typical control specifications are:

- ▶ steady state error (tracking accuracy)
- ▶ overshoot, or damping ratio, or phase margin (oscillation)

[B747-100] Investigate the open-loop frequency response by the Bode plot, again taking the negative $-G_{\theta e}(j\omega)$,



(a) exact



(b) phugoid

1.1 PID Control

1. Proportional (P) Control
2. Proportional-Derivative (PD) Control
3. Proportional-Integral (PI) Control
4. Proportional-Integral-Derivative (PID) Control

► Proportional Control

$$u(t) = K_p e(t)$$

$$U(s) = K_p E(s)$$

► Proportional-Derivative Control

$$u(t) = K_p e(t) + K_d \dot{e}(t)$$

$$U(s) = (K_p + K_d s) E(s)$$

► Proportional-Integral Control

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau) d\tau$$

$$U(s) = \left(K_p + \frac{K_i}{s} \right) E(s)$$

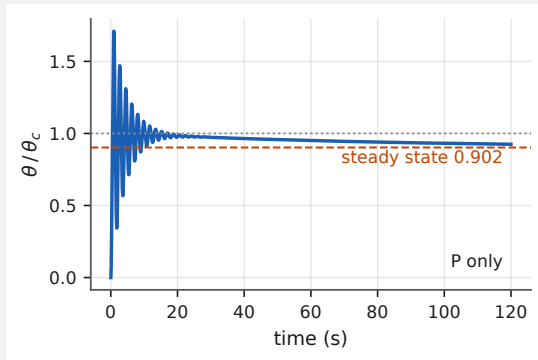
► PID Control

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau) d\tau + K_d \dot{e}(t)$$

$$U(s) = \left(K_p + \frac{K_i}{s} + K_d s \right) E(s)$$

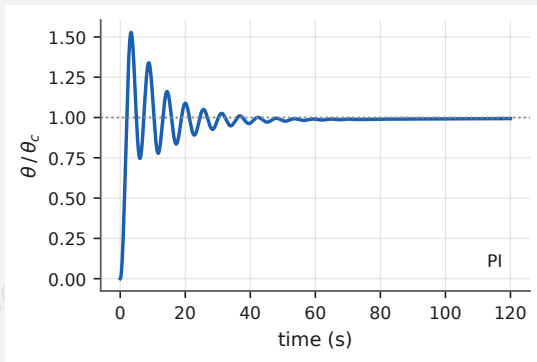
The proportional-integral-derivative control method:

$\delta_e = -K_p e_\theta - K_i \int e_\theta dt - K_d \dot{e}_\theta$ focuses on tuning the PID gains K_p , K_i , and K_d . Let's say we require that the steady-state error to a pitch attitude command stays within 10%. Back to the same numerical case study, it leads to a desired gain $K_p = 10$.



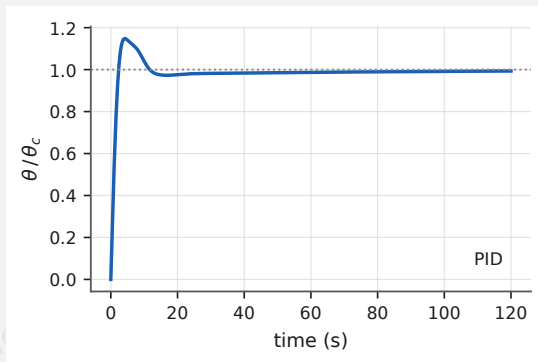
Pitch PID Control: $K_p = -10$

It suggests that the satisfied steady-state error also sacrifices with large oscillation. The single gain treatment doesn't get a good tradeoff. The PI control, with smaller gains, seems to get similar response.



Pitch PID Control: $K_p = -0.5$, $K_i = -0.5$

On the other hand, if one properly tunes all PID gains, such as the following design, we may get good performance.



Pitch PID Control: $K_p = -0.5$, $K_i = -0.5$, $K_d = -0.5$

How to tune PID gains is an art and an on-going study. A systematic alternative to trial and error is to decide first where the closed-loop poles should lie, then solve for the gains that put them there.

Write the plant as $P(s) = N(s)/D(s)$ and the controller as

$$C(s) = K_p + K_i \frac{1}{s} + K_d s = \frac{K_d s^2 + K_p s + K_i}{s}.$$

The closed-loop characteristic equation is then

$$sD(s) + (K_d s^2 + K_p s + K_i) N(s) = 0.$$

This expression is *linear* in K_p , K_i and K_d . Requiring it to vanish at three chosen values of s therefore gives three linear equations in the three gains.

Design by dominant-pole specification:

- ▶ Choose a dominant pair $s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$ from the transient requirement: overshoot sets ζ , speed of response sets ω_n .
- ▶ Choose one further pole s_3 , far enough to the left that the pair dominates.
- ▶ Impose the characteristic equation at s_1 , s_2 and s_3 , and solve the resulting 3×3 linear system for K_d , K_p and K_i .
- ▶ Examine the poles that were *not* assigned.

A PID has three gains, so for an n th-order plant the closed loop is of order $n + 1$ and only three of its poles may be assigned. The remaining $n - 2$ follow from the algebra and must be verified rather than assumed.

Choosing the dominant pair. The two numbers answer separate requirements: ζ follows from the permitted overshoot, ω_n from the required speed of response,

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}, \quad \zeta = \frac{-\ln M_p}{\sqrt{\pi^2 + \ln^2 M_p}}, \quad t_s \simeq \frac{4}{\zeta\omega_n} \quad (2\%)$$

ζ	0.4	0.5	0.6	0.7	0.8
M_p	25.4%	16.3%	9.5%	4.6%	1.5%

On the s -plane the two act in independent directions: ζ fixes the angle $\arccos \zeta$ measured from the negative real axis, ω_n fixes the distance from the origin, and their product $\zeta\omega_n$ – the real part – fixes the decay rate. A transient specification is therefore a wedge intersected with a half-plane, and any point inside it will serve.

For an aircraft the pair is not a free choice. Handling-qualities requirements bound the short-period damping; the Level 1 limits are

A, C — manoeuvring, takeoff, landing	$0.35 \leq \zeta_{sp} \leq 1.30$
B — climb, cruise, descent	$0.30 \leq \zeta_{sp} \leq 2.00$

They bound ζ rather than determine it: Level 1 admits overshoot anywhere from zero to some 30%. The transient specification picks the value inside, and $\zeta = 0.7$ is the natural choice – it gives 4.6% overshoot and lies at the geometric centre of the A/C range, $\sqrt{0.35 \times 1.30} = 0.67$, a factor of two from either limit. The frequency follows from $CAP = \omega_{nsp}^2 / (n/\alpha)$, with ω_n limited in practice by actuator bandwidth and rate saturation.

The third pole s_3 carries no specification, but must be real: the gains are real, so a complex choice would drag in its conjugate, imposing a fourth condition on three gains. It is placed far enough left – five times the real part of the pair is a common rule – that the pair remains dominant.

The design equations. Evaluating the characteristic equation at a chosen pole s_k and dividing by $N(s_k)$ isolates the gains,

$$K_d s_k^2 + K_p s_k + K_i = -\frac{s_k D(s_k)}{N(s_k)} = -\frac{s_k}{P(s_k)} \equiv b_k.$$

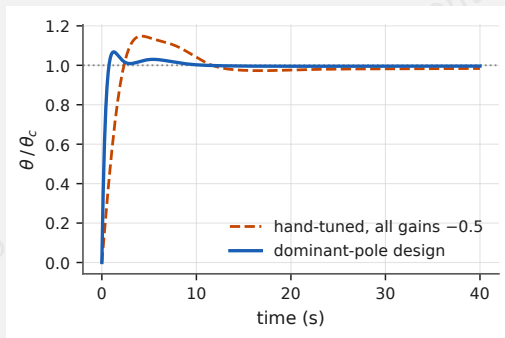
The three conditions assemble into

$$\begin{bmatrix} s_1^2 & s_1 & 1 \\ s_2^2 & s_2 & 1 \\ s_3^2 & s_3 & 1 \end{bmatrix} \begin{bmatrix} K_d \\ K_p \\ K_i \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

a Vandermonde system, non-singular whenever s_1 , s_2 and s_3 are distinct. Since N and D are real, $b_2 = \bar{b}_1$ and the second row is the conjugate of the first. Taking the real and imaginary parts at s_1 , together with the condition at the real pole s_3 , leaves a *real* linear system of three equations in the three real unknowns K_d , K_p , K_i . No s_k may be a zero of N : there the equation collapses to $s_k D(s_k) = 0$, which no gains can satisfy.

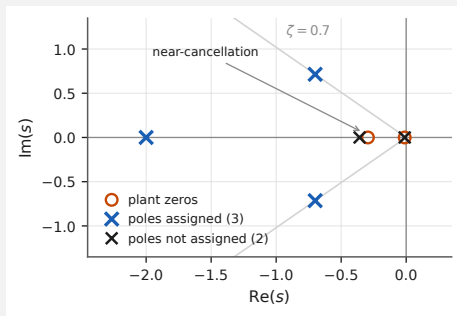
[Example 1] For the B747-100 pitch channel, take $\zeta = 0.7$ and $\omega_n = 1$ rad/s, with the third pole at $s_3 = -2$. Solving the linear system gives

$$K_p = -2.769, \quad K_i = -2.091, \quad K_d = -2.610.$$



Overshoot 14.8% $\rightarrow$ 6.8%; 2% settling 23.5 s $\rightarrow$ 7.2 s

The two unassigned poles are inspected before the design is accepted.



They lie at -0.358 and -0.0113 , close to the plant zeros at -0.294 and -0.0114 . Each is nearly cancelled by the neighbouring zero and so contributes little to the response. This migration of closed-loop poles towards the open-loop zeros is a general feature of increasing loop gain, and is taken up in the root-locus construction that follows.

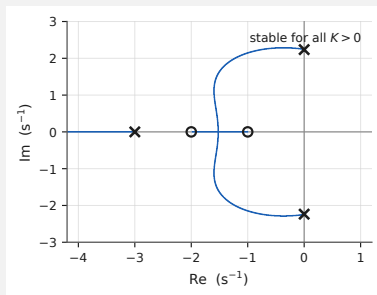
1.2 Root Locus Design and Analysis

- ▶ Closed-loop poles depend on K
- ▶ Root locus is a plot showing how the roots of $D(s) + KN(s) = 0$ vary with $K, 0 \leq K < \infty$

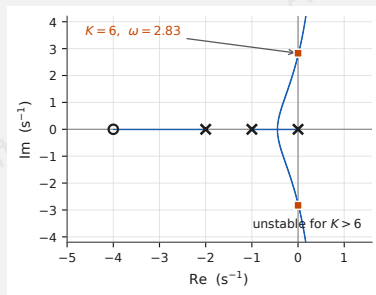
Rules of Constructing Root Locus

- ▶ Main rules are established by theorems (with proof)
- ▶ Corresponding steps in plot construction is numbered.

[Example 2] Two loci showing the basic features of the construction.



(a) $G(s) = \frac{(s+1)(s+2)}{(s+3)(s^2+5)}$



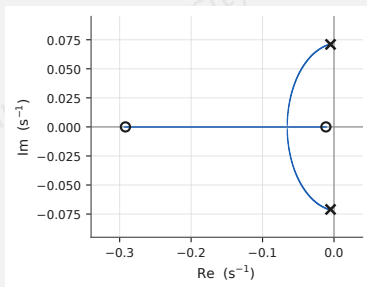
(b) $G(s) = \frac{s+4}{s(s+2)(s+1)}$

Question 1. What can you tell about stability?

The phugoid mode dominates, so the locus is drawn for the phugoid approximation of the pitch channel. Closing the loop as $\delta_e = -K(\theta_c - \theta)$ makes the design plant $P_\theta = -G_{\theta e}$, of positive DC gain, and the closed-loop roots solve $D(s) + KN(s) = 0$ with

$$P_\theta^{pg}(s) = \frac{1.4114(s + 0.2919)(s + 0.0113)}{(s + 0.0048)^2 + 0.0710^2}$$

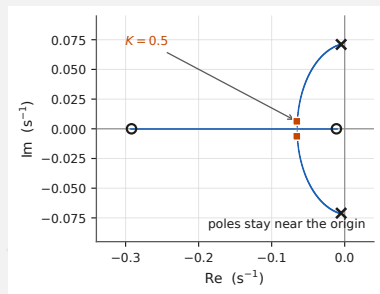
and $P_\theta^{pg}(0) = 0.923$.



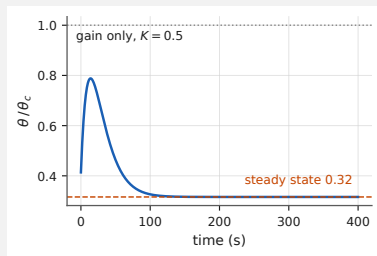
A single gain cannot satisfy both requirements. Accuracy demands a large K , while the phase margin of the full aircraft demands a small one:

K	0.5	1	2	5	10
error	68%	52%	35%	18%	10%
PM	119°	40°	23°	12°	8°

At $K = 0.5$ the response peaks at 0.79 and settles at 0.32.



(a) design



(b) result

Both branches leave the phugoid poles and end on the plant zeros at -0.2919 and -0.0113 , so no K moves a root further left than -0.29 : the loop can be made accurate or well damped, never fast. With $P_{\theta}^{pg} = N/D$ as above, the characteristic equation is

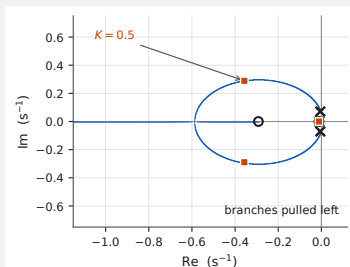
$$D(s) + KN(s) = 0,$$

and inserting the compensator $C(s) = K/(s + 0.01)$ changes it to

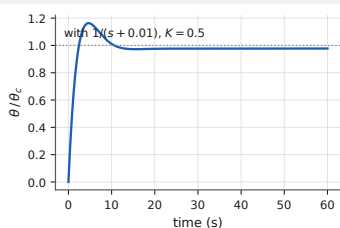
$$(s + 0.01) D(s) + KN(s) = 0,$$

a cubic rather than a quadratic, so a third branch appears. The new pole lies within 12% of the zero at -0.0113 : that pair nearly cancels and the remaining branches swing far to the left. The same pole raises the DC loop gain by $1/0.01 = 100$, so at unchanged $K = 0.5$ the steady-state error falls from 68% to 2%.

The resulting locus and closed-loop response:



(a) design



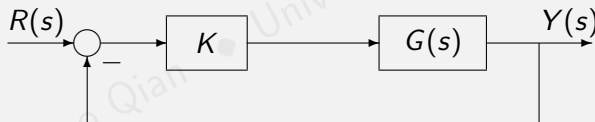
(b) result

Both plots use the phugoid approximation. On the full fourth-order model this compensator is *unstable*: $1/(s + 0.01)$ attenuates nothing at the short-period frequency but contributes -89° of phase, dragging the crossover from 0.29 to 0.90 rad/s and the phase margin from $+119^\circ$ to -8° . The approximation cannot reveal this, having no short period to lose. The lag compensator of the next section adds a zero to restore that phase.

1.3 Lead Compensator Design

Let's review the frequency response and design specifications. The plots of

- ▶ the magnitude $|G(j\omega)|$ ($20 \log_{10} |G(j\omega)|$) vs ω , and
- ▶ the phase angle $\arg G(j\omega)$ vs ω



Consider the Bode plot of the *open-loop* system $KG(s)$,

- ▶ gain crossover frequency ω_g , where $|KG(j\omega)| = 1$, or 0 dB
- ▶ phase crossover frequency ω_π , where $\arg KG(j\omega) = -180^\circ$

the stability margins for the *closed-loop* system:

- ▶ gain margin, read at the *phase* crossover, $GM = \left| \frac{1}{KG(j\omega_\pi)} \right|_{dB}$,
positive when $|KG(j\omega_\pi)|_{dB} < 0$ dB
- ▶ phase margin, read at the *gain* crossover, $PM = 180^\circ + \arg KG(j\omega_g)$, positive when $\arg KG(j\omega_g) > -180^\circ$

$$C(s) = K \frac{Ts + 1}{\alpha Ts + 1}, \quad 0 < \alpha < 1$$

The impact of the dynamic component $\frac{Ts+1}{\alpha Ts+1}$ is seen in its Bode plot: phase is added (hence *lead* compensator) and the magnitude rises. At the pole corner $\omega = 1/\alpha T$ it has reached

$$\left| \frac{j\omega T + 1}{j\alpha\omega T + 1} \right|_{\omega=1/\alpha T} = \sqrt{\frac{1 + 1/\alpha^2}{2}} \simeq \frac{1}{\alpha\sqrt{2}},$$

that is 3 dB below its final value, and only for $\omega \gg 1/\alpha T$ does it approach the asymptote

$$\left| \frac{j\omega T + 1}{j\alpha\omega T + 1} \right|_{dB} \rightarrow 20 \log \frac{1}{\alpha}$$

The maximum phase increase $\phi_{\max}$ happens at the frequency

$$\omega_{\max} = \frac{1}{\sqrt{\alpha} T}$$

Also, we note that

$$\alpha = \frac{1 - \sin \phi_{\max}}{1 + \sin \phi_{\max}}$$

and when $\omega = \omega_{\max}$, the dynamic compensator brings increase of magnitude

$$\left| \frac{j\omega_{\max} T + 1}{j\alpha\omega_{\max} T + 1} \right|_{dB} = 20 \log \frac{1}{\sqrt{\alpha}}$$

$$\begin{aligned}\omega_{max} &= \frac{1}{\sqrt{\alpha}T} \\ \phi_{max} &= \tan^{-1} \frac{1}{\sqrt{\alpha}} - \tan^{-1} \sqrt{\alpha} \\ \sin \phi_{max} &= \frac{1 - \alpha}{1 + \alpha}, \text{ or } \alpha = \frac{1 - \sin \phi_{max}}{1 + \sin \phi_{max}} \\ |C(j\omega_{max})| &= \frac{K}{\sqrt{\alpha}}\end{aligned}$$

Note which quantity each statement refers to: the *dynamic factor* $\frac{T_s+1}{\alpha T_s+1}$, or the whole compensator $C = K \frac{T_s+1}{\alpha T_s+1}$.

	dynamic factor	whole C
$\omega \rightarrow 0$	1	K
$\omega = \omega_{max}$	$1/\sqrt{\alpha}$	$K/\sqrt{\alpha}$
$\omega \rightarrow \infty$	$1/\alpha$	K/α

- ▶ the dynamic factor has DC gain exactly 1, hence $C(0) = K$: the steady-state design of step 1 is untouched by T and α
- ▶ high-frequency gain is raised by the factor $1/\alpha$
- ▶ phase is added in the crossover region

The design concept is to maximize the phase margin of $C(j\omega)G(j\omega)$ by setting the maximum phase $\phi_{\max}$ around a critical frequency.

- ▶ This critical frequency should be around the uncompensated plant $KG(j\omega)$ gain cross-over frequency (remember, we care of the frequency range of $0 \sim \omega_{gk}$), but choose a slightly different frequency point $\omega_{\max}$ (of course to provide maximum phase increase by lead compensator)
- ▶ and this frequency shall also be the point where uncompensated $KG(j\omega)$ may have a negative magnitude value that is expected to be compensated by the increase of magnitude introduced by lead compensator at $\omega_{\max}$

- ▶ How much? If lead compensator brings increase of magnitude by $20 \log \frac{1}{\sqrt{\alpha}}$ at $\omega_{\max}$, as above, we shall find this point where current uncompensated one has a negative value of $-20 \log \frac{1}{\sqrt{\alpha}}$. This way, we expect the $\omega_{\max}$ happens to be the gain cross-over frequency ω_{gh} (g - gain cross-over frequency, h - the newly designed gain cross-over frequency under dynamic controller C instead of K).

After the design, we acknowledge that when $\omega = \omega_{gh} = \omega_{\max}$

$$|C(j\omega_{gh})G(j\omega_{gh})| = |KG(j\omega_{gh})| \left| \frac{j\omega_{\max}T + 1}{j\alpha\omega_{\max}T + 1} \right| = \sqrt{\alpha} \frac{1}{\sqrt{\alpha}} = 1$$

and phase here should give the desired phase margin now.

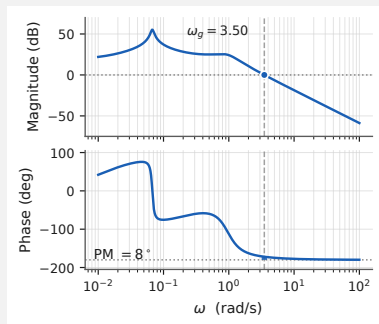
When $\omega \gg 1/\alpha T$, we do have an increase of magnitude, as noted above,

$$|C(j\omega)G(j\omega)| = |KG(j\omega)| \left| \frac{j\omega T + 1}{j\alpha\omega T + 1} \right| = |KG(j\omega)| \frac{1}{\alpha} = |K_1 G(j\omega)|$$

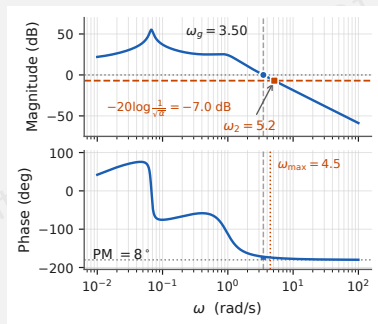
But the good news is, we don't care any more for the large frequency $1/\alpha T$ as we have designed to meet the specifications for the dynamic performance with the proper range of $0 \sim \omega_{gh}$.

Design process (graphical measurement is underlined)

1. Design K to give desired steady-state value
2. Draw Bode Plot of uncompensated $KG(j\omega)$
3. measure the uncompensated gain cross-over frequency $\omega_1 = \omega_{gk}$ and measure its phase margin PM_k
4. Determine ϕ_{max} which must be added to obtain the desired PM (allow for extra $5 \sim 15^\circ$), solve for α
5. find a critical frequency ω_2 where $|KG(j\omega_2)|_{dB} = -20 \log \frac{1}{\sqrt{\alpha}}$. This ω_2 corresponds to the $\omega_{max} = \frac{1}{\sqrt{\alpha}T}$, solve for T . This ω_2 is also the expected ω_{gh} .
6. Form $C(s) = K \frac{T_s+1}{\alpha T_s+1}$. Draw Bode plot of $H(j\omega)G(j\omega)$ for verification



(a) steps 2–3: uncompensated KG ,
 $\omega_{gk} = 3.50$, $PM_k = 8^\circ$



(b) steps 4–5: $|KG| = -7.0$ dB at
 $\omega_2 = 5.2$, so $\alpha = 0.2$, $T = 0.5$

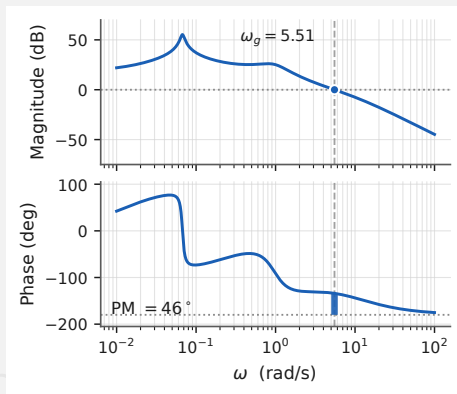
Before: the uncompensated loop and the graphical construction

Worked Example: Steps 1–5 for the B747-100

1. $P_\theta(0) = 0.923$. For $e_{ss} = \frac{1}{1+KP_\theta(0)} \leq 10\%$, $K = \frac{1/0.1-1}{0.923} = 9.75$; take $K = 10$, which gives 9.8%.
2. Draw the Bode plot of $KG(j\omega)$ — figure (a).
3. **measure** $\omega_{gk} = 3.50$ rad/s and $\text{PM}_k = 8^\circ$.
4. Target 45° with a 5° allowance: $\phi_{\max} = 45 - 8 + 5 = 42^\circ$, hence $\alpha = \frac{1-\sin \phi_{\max}}{1+\sin \phi_{\max}} = 0.20$.
5. At $\omega_{\max}$ the lead lifts the curve by $20 \log \frac{1}{\sqrt{\alpha}} = 7.0$ dB, so **find** where the uncompensated $|KG| = -7.0$ dB: $\omega_2 = 5.2$ rad/s — figure (b). Then $T = \frac{1}{\omega_2 \sqrt{\alpha}} = 0.43$, rounded to 0.5.

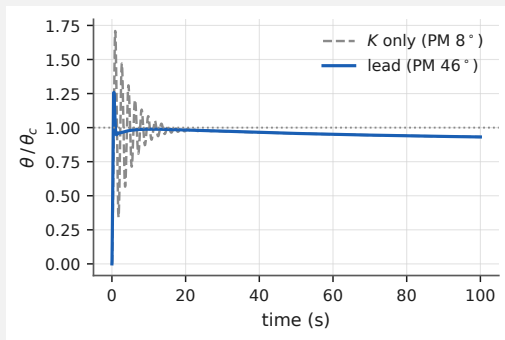
$$C(s) = 10 \frac{0.5s + 1}{0.1s + 1}$$

Rounding T to 0.5 moves $\omega_{\max} = \frac{1}{T\sqrt{\alpha}}$ from 5.2 to 4.5, which is why figure (b) carries both labels. Step 6 then verifies $\omega_g = 5.51$ rad/s, $\text{PM} = 46^\circ$.



After: $C(j\omega)G(j\omega)$ with $C(s) = 10 \frac{0.5s+1}{0.1s+1}$

Crossover moves $3.50 \rightarrow 5.51$ rad/s, phase margin $8^\circ \rightarrow 46^\circ$



Closed-loop step response before and after the lead compensator

1.4 Lag Compensator

Why a lag. K is already fixed by the steady-state requirement and cannot be lowered, yet the phase margin at ω_{gk} is poor. Lower down in frequency the plant's phase is better; all that is missing is for the crossover to be *there*.

On the [B747-100] pitch channel the bind is explicit:

	phase margin	steady-state error
$K = 0.9$	44°	55%
$K = 10$	8°	10%

The margin we want is reachable by gain alone, at $K \simeq 0.9$ — but the accuracy specification forbids it. A lag attenuates around the crossover while leaving the DC gain at K : it lowers the gain *without lowering K* .

Properties of the Lag Compensator

$$C(s) = K \frac{Ts + 1}{\beta Ts + 1}, \quad \beta > 1$$

Zero at $-1/T$, pole at $-1/(\beta T)$. Since $\beta > 1$ the *pole* is nearer the origin — the mirror image of the lead.

	dynamic factor	whole C
$\omega \rightarrow 0$	1	K
$\omega \rightarrow \infty$	$1/\beta$	K/β

At the zero corner $\omega = 1/T$ the factor has reached

$$\left| \frac{j\omega T + 1}{j\beta\omega T + 1} \right|_{\omega=1/T} = \sqrt{\frac{2}{1 + \beta^2}} \simeq \frac{\sqrt{2}}{\beta},$$

that is 3 dB *above* its final value $20 \log \frac{1}{\beta}$, which it reaches only for $\omega \gg 1/T$. The phase is negative everywhere (hence *lag*) but decays to almost nothing there — -5° at $\omega = 10/T$.

How the Lag Achieves It

The property that makes it work. For $\omega \gg 1/T$,

$$\left| \frac{j\omega T + 1}{j\beta\omega T + 1} \right| \simeq \frac{1}{\beta}, \quad \text{with almost no phase change.}$$

So choosing $1/T \ll \omega_{gk}$ gives pure attenuation and leaves the phase shape intact.

The construction.

- ▶ On the *uncompensated* plot, locate ω_{gh} , the frequency where the phase already provides the desired margin.
- ▶ Read the magnitude there and set $\beta = |KG(j\omega_{gh})|$.
- ▶ Then $|C(j\omega_{gh})G(j\omega_{gh})| = \beta \cdot \frac{1}{\beta} = 1$, so ω_{gh} becomes the new crossover — carrying the margin that was chosen when it was located.

Elsewhere. At the old crossover $|CG(j\omega_{gk})| = 1/\beta$, well below 1, so ω_{gk} no longer governs; above it everything falls by $1/\beta$, exactly as a reduced gain K/β would. DC is untouched, so the steady-state design survives.

Design process (graphical measurement is underlined)

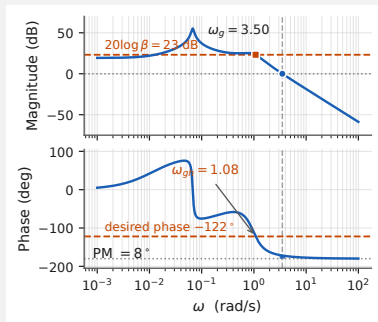
1. Design K to give desired steady-state value
2. Draw Bode Plot of $KG(j\omega)$
3. measure the uncompensated gain cross-over frequency $\omega_1 = \omega_{gk}$ and measure its phase margin PM_k - for info only.
4. find a desired phase based on desired phase margin (allow for extra $5 \sim 15^\circ$ margin) and locate its corresponding frequency ω_2 . This is the expected ω_{gh} . measure $|KG(j\omega_2)|_{dB}$
5. Place zero of the compensator such that $1/T$ is below ω_2 by an octave to by a decade, solve for T .
6. Evaluate $|KG(j\omega_2)|_{dB} = 20 \log \beta$. Solve for β .
7. Form $C(s) = K \frac{Ts+1}{\beta Ts+1}$. Draw Bode plot of $C(j\omega)G(j\omega)$ for verification

Worked Example: Lag Design, Steps 1–6

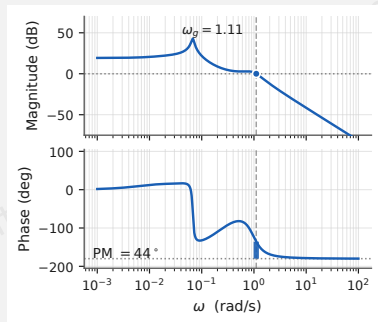
1. Same steady-state requirement as before, so the same $K = 10$.
2. Draw the Bode plot of $KG(j\omega)$.
3. **measure** $\omega_{gk} = 3.50$ rad/s, $PM_k = 8^\circ$ — for information only; the lag will not use this crossover.
4. Target 45° with a 13° allowance, so **find** the phase $-180^\circ + 58^\circ = -122^\circ$ and **locate** it: $\omega_2 = 1.08$ rad/s. There **measure** $|KG(j\omega_2)| = 23$ dB.
5. Place $1/T$ below ω_2 by an octave to a decade: $T = 4.5$ gives $1/T = 0.22 = \omega_2/4.9$.
6. $20 \log \beta = 23$ dB, so $\beta = 14.4$; take $\beta = 14$.

$$C(s) = 10 \frac{4.5s + 1}{63s + 1}$$

Step 7 verifies $\omega_g = 1.11$, $PM = 44^\circ$. The allowance pays for the lag's own phase at the new crossover, -10.5° ; this is also why $1/T$ sits well below ω_2 — at ω_2 it would cost -41° .



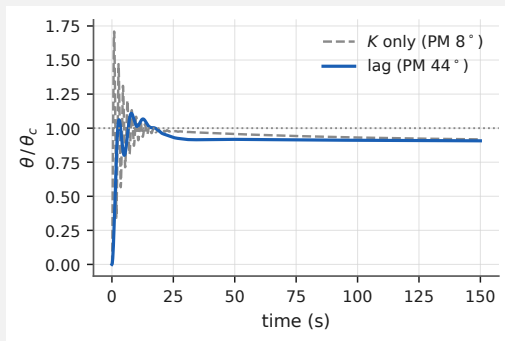
(a) before: -122° at $\omega_g = 1.08$,
 $20 \log \beta = 23$ dB



(b) after: $\omega_g = 1.11$, $PM = 44^\circ$

Before and after: $C(s) = 10^{\frac{4.5s+1}{63s+1}}$, $\beta = 14$

Crossover moves $3.50 \rightarrow 1.11$ rad/s, phase margin $8^\circ \rightarrow 44^\circ$



Closed-loop step response before and after the lag compensator
The lead raised the crossover; the lag lowers it, buying margin with speed

Lead or lag?

Lead $K \frac{T_s+1}{\alpha T_s+1}$, $\alpha < 1$: adds phase, at most $\sin \phi_{\max} = \frac{1-\alpha}{1+\alpha}$, and lifts the gain above $1/T$. The crossover *rises* — faster, but with $1/\alpha$ more high-frequency gain, so more sensor noise and actuator activity.

Lag $K \frac{T_s+1}{\beta T_s+1}$, $\beta > 1$: attenuates by $1/\beta$ above $1/T$, raising the DC gain by β . The crossover *falls* — slower, but quieter and more accurate. Keep the zero well below the new crossover so its own lag does not consume the margin gained.

Use lead when the loop needs margin *and* bandwidth; lag when accuracy is the problem, or bandwidth may be traded for margin.

	ω_g	PM	90% rise	overshoot	2% settling
lead	5.51	46°	0.25 s	26%	129 s
lag	1.11	44°	1.85 s	11%	28 s

Both reach $\approx 45^\circ$: the lead rises six times faster but settles later, each leaving a pole near the plant zero at -0.0113 , the lead's further off.

1.5 Pitch Rate Control with Actuation

An inner pitch-rate loop is usually closed first, and the elevator servo is no longer ideal: it contributes its own lag $G_a(s)$ between the commanded and the achieved deflection.

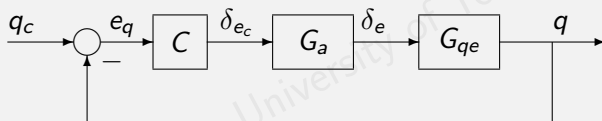
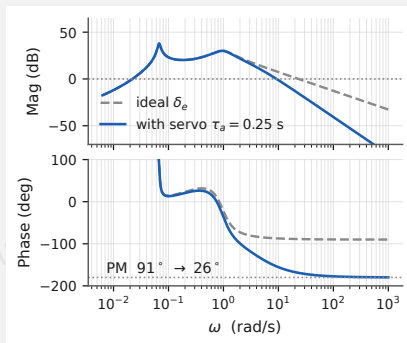


Figure 2: Pitch Rate Control with Elevator Actuation

A first-order servo $G_a(s) = 1/(\tau_a s + 1)$ adds phase lag near the gain crossover, so the margins measured on G_{qe} alone are optimistic.

What the Servo Costs

With $\tau_a = 0.25$ s and a proportional gain $K = 20$, the servo lag removes most of the available margin without changing the plant at all:

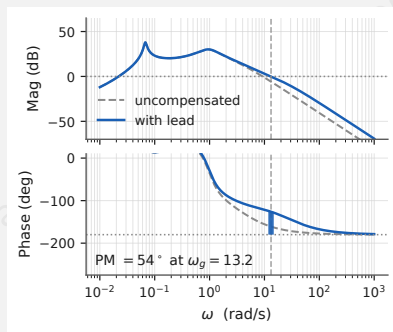


The crossover falls from 23.2 to 9.3 rad/s and the phase margin from 91° to 26° :
designing on G_{qe} alone predicts a margin the real loop never has.

Pitch Rate: Lead Compensation

A lead restores the phase at crossover. Taking $\phi_{\max} = 35^\circ$ gives $\alpha = 0.27$, and placing $\omega_{\max}$ at the -5.7 dB point gives $T = 0.15$:

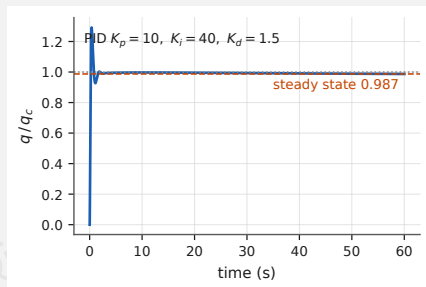
$$C(s) = 20 \frac{0.15s + 1}{0.0405s + 1}$$



The margin recovers to 54° and the crossover rises to 13.2 rad/s — above the uncompensated value, not merely restored.

Pitch Rate: PID and the Type of the Loop

Lead fixes the margin but not the tracking: $G_{q\delta_e} = sG_{\theta\delta_e}$ carries a zero at $s = 0$, so a proportional or lead controller returns $q \rightarrow 0$ for any step command. Integral action is required.



$K_p = 10$, $K_i = 40$, $K_d = 1.5$ gives $PM = 50^\circ$ and settles in 26 s. The integrator is *consumed* cancelling the plant zero, so the loop is still type 0 in q : the steady value is $K_i G'(0)/(1 + K_i G'(0)) = 0.987$, not 1. A sustained pitch rate is a ramp in θ , which no finite elevator deflection can hold.

2. Speed Control

Technically speaking, from the longitudinal linear equations and the corresponding transfer functions, the speed u is influenced by both the throttle δ_p and the elevator δ_e channels in different ways. As discussed previously, the throttle advancement will enable speed response initially but little settling effect (the thrust contributes to the steady climb instead); and the elevator influences the speed variation through the pitch angle control (from the kinetic and potential energy exchange perspective). Nevertheless, we have choices of selecting either channel to regulate the speed through automatic feedback loop.

Question 2. How would you approach speed control?

Speed control through δ_p

Similar to the pitch attitude control channel, the speed control loop through δ_p is illustrated in Fig. 3. It is straightforward, by following classical control design approaches, we will try to find the proper control laws for speed regulation.

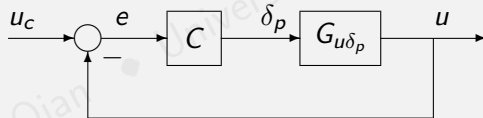


Figure 3: Speed Control through δ_p

Again, the B747 case example gives

$$G_{up} = \frac{2.943s^3 + 2.187s^2 + 2.73s}{s^4 + 0.7499s^3 + 0.9341s^2 + 0.009449s + 0.004187}$$

where the open-loop zeros and poles are:

$$z_{1,2} = -0.3715 \pm 0.8887i, z_3 = 0,$$

$$p_{1,2} = -0.3717 \pm 0.8869i, p_{3,4} = -0.0033 \pm 0.0672i.$$

The phugoid approximation gives the following analytical expression of the transfer function:

$$G_{u\delta_p}^{ph} = \frac{b_1 s}{As^2 + Bs + C}$$

where

$$b_1 = -U_e M_w \frac{X_{\delta_p}}{m}$$

Taking further approximations of A, B, C as we did before, we have the following expression:

$$\begin{aligned} G_{u\delta_p}^{ph} &\approx \frac{-U_e M_w \frac{X_{\delta_p}}{m} s}{-U_e M_w s^2 + \frac{U_e}{m} X_u M_w s + \frac{g}{m} Z_u M_w} \\ &= \frac{\frac{X_{\delta_p}}{m} s}{s^2 - \frac{X_u}{m} s - \frac{g}{m} \frac{Z_u}{U_e}} \end{aligned}$$

In [B747-100] case, the phugoid mode approximation becomes

$$G_{u\delta_p}^{ph} = \frac{2.943s}{s^2 + 0.009685s + 0.005065}$$

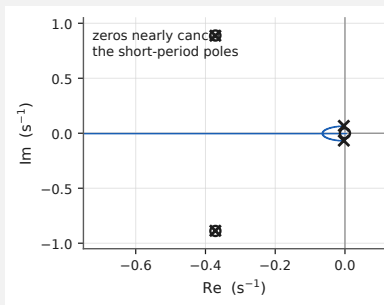
also generating the open-loop zeros and poles: $z_1 = 0$,
 $p_{1,2} = -0.0048 \pm 0.0710i$.

From the transfer function $G_{u\delta_p}$, we can tell that a step input of throttle will cause transient speed response but no steady value variation. On the other hand, if one has a steady acceleration of throttle, in other words, the δ_p is a ramp function, then we shall expect non-zero steady increase of speed, approximately by the value of

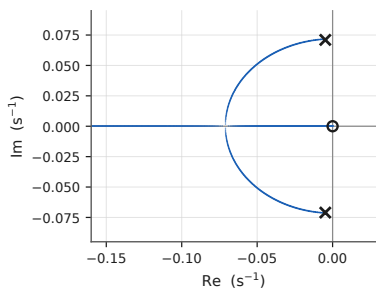
$$u_{ss, \delta_p=1/s^2} = -\frac{X_{\delta_p} U_e}{gZ_u} \approx -0.3 \frac{mU_e}{Z_u}$$

in our [B747-100] case.

The root-locus plots may not help to determine the control gain, but reveal that the cancelling effect of zeros and short-period poles of the transfer function.

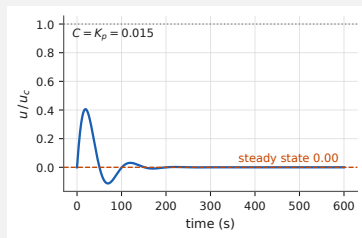


(a) $G_{u\delta_p}$

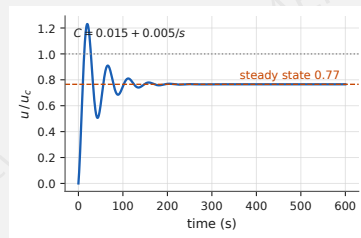


(b) $G_{u\delta_p}^{ph}$

Closing the Throttle Loop: What the Zero at the Origin Costs



(a) $C = K_p = 0.015$



(b) $C = 0.015 + 0.005/s$

$G_{u\delta_p}$ carries a zero at $s = 0$, so $G_{u\delta_p}(0) = 0$ and proportional control gives $e_{ss} = 1$: 100% **error for any gain**, as (a) shows.

Integral action helps but does not cure it — the plant zero at the origin *cancels* the controller's integrator, so the loop stays type 0 and $L(0)$ is finite. In (b) that leaves 23% error; raising K_i to 0.01 and 0.1 gives 13% and 1.5%. Throttle alone commands climb rate, not speed: holding a speed needs the attitude loop closed first.

Speed control through δ_e

One may wonder, what about the speed control by the elevator? After all, it effectively controls the pitch angle, in other words, the potential energy. Therefore, it should lead to the regulation to the kinetic energy, i.e. speed. In fact, the speed control through the elevator shall be investigated, with some special care.

When we look at the phugoid approximation of $G_{u\delta_e}^{pg}$, in the [B747-100] case, we have the following expression

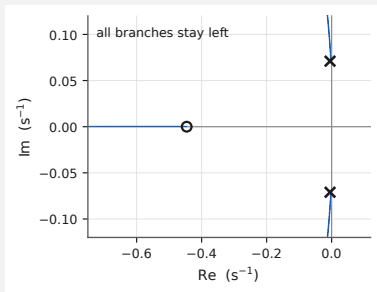
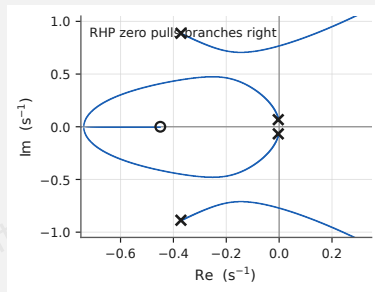
$$G_{u\delta_e}^{ph} = \frac{9.204s + 4.104}{s^2 + 0.009685s + 0.005065}$$

where the poles $p_{1,2} = -0.0048 \pm 0.0710i$ and the zero of $z = -0.4459$ appropriately represent the phugoid mode. However, the full $G_{u\delta_e}$

$$G_{u\delta_e} = \frac{-5.726e - 05s^3 - 0.07684s^2 + 7.511s + 3.392}{s^4 + 0.7499s^3 + 0.9341s^2 + 0.009449s + 0.004187}$$

revealing zeroes: $-1433.3, +91.9, -0.4$. Apparently, one significant information is missing, that is the positive zero $z = 91.9$. In other words, the non-minimum phase impact of $G_{u\delta_e}$ is not explicit from the phugoid approximation.

First of all, the difference can be shown from their respective root locus plot:

(a) $G_{u\delta_e}^{ph}$, phugoid approximation(b) $G_{u\delta_e}$, full model

The approximation keeps every branch in the left half-plane, suggesting the loop is stable for any K . The full model carries the zero at $+91.9$, and branches terminate on zeros, so one of them must finish in the right half-plane: the closed loop turns unstable at $K = 0.058$, crossing the imaginary axis at $\pm 0.769j$.

It is clear that the non-minimum phase system, with a positive zero on the right half plane, tends to 'pull' the closed-loop poles to the right, even lead to unstable results. Therefore, it is obvious that one shall be careful when choosing the gain values (not to cross the imaginary axis).

There is more to the story of the non-minimum phase system. It also reveals an undesired negative initial response.

$$\begin{aligned}u(0^+) &= \lim_{s \rightarrow \infty} s G_{u\delta_e} \frac{1}{s} = 0 \\ \dot{u}(0^+) &= \lim_{s \rightarrow \infty} s \left(s G_{u\delta_e} \frac{1}{s} - u(0^+) \right) < 0\end{aligned}$$

In the [B747-100] case, the negative value is small enough that the impact is benign, $\dot{u}(0^+) = -0.000057$. But the trend can be shown from the short period movement of w . In other words, in order to increase the speed, due to the non-minimum phase, the aircraft experiences a drop of speed first before picking up. It may not be desirable.

Summary: Classical Design on the Pitch Channel

All designs below use the same plant $-G_{\theta\delta_e}$ and the same static gain $K = 10$, fixed by the 10% steady-state requirement.

design	ω_g (rad/s)	PM	overshoot	2% settling
K only, $K = 10$	3.50	8°	71%	128 s
PID $-0.5, -0.5, -0.5$	0.83	71°	15%	23 s
lead $10 \frac{0.5s+1}{0.1s+1}$	5.51	46°	26%	129 s
lag $10 \frac{4.5s+1}{63s+1}$	1.11	44°	11%	28 s
root locus $+1/(s+0.01)^\dagger$	—	—	16%	11 s

[†] designed on the phugoid approximation; all other rows use the full model.

- ▶ Lead and lag reach the *same* margin by opposite routes: lead raises the crossover, lag lowers it.
- ▶ A margin fixes the transient, not the tail — the slow phugoid pole survives any compensator acting near 5 rad/s.
- ▶ Integral action (PI, or the near-integrator $1/(s+0.01)$) is what removes steady-state error; K alone cannot.

Summary: Speed Channel and What to Watch For

Throttle channel $G_{u\delta_p}$

- ▶ A zero at $s = 0$ forces 100% steady-state error under proportional control, at *any* gain. Integral action is necessary but *not sufficient*: that same zero cancels the controller's integrator, leaving 23% error at $K_i = 0.005$.
- ▶ Its zeros nearly cancel the short-period poles, so the root locus is governed by the phugoid.
- ▶ Proportional and PI designs for this channel are worked in Tutorial 2.

Elevator channel $G_{u\delta_e}$

- ▶ Non-minimum phase: zeros at -1433 , $+91.9$, -0.45 .
- ▶ The right-half-plane zero pulls branches right; the loop goes unstable near $K \approx 0.06$.
- ▶ The phugoid approximation *hides* this zero — use the full model for any frequency-domain or gain-limit work.
- ▶ Speed drops before it rises, $\dot{u}(0^+) < 0$.

Reduced models are for insight; verify every design on the full model.

Neither channel alone will do. Close the pitch loop with the elevator, then command speed with the throttle — the architecture taken up in Lecture 05.

Summary

① Pitch Attitude Control

② Speed Control