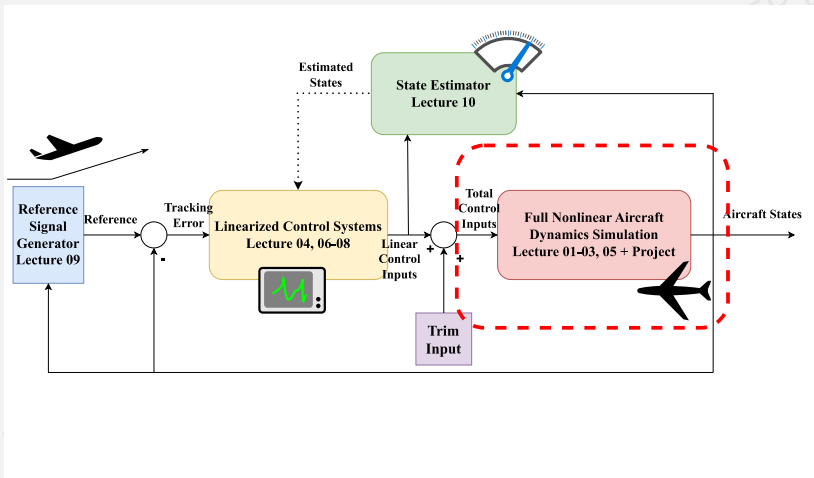


Lecture 03

Flight Stability and Control Using Linearized Aircraft Model

September 30, 2026

Acknowledgement: These slides were developed with reference to course materials by Professor Hugh H. T. Liu.



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1. A Brief Review of Linear Systems

1. **Linearized aircraft models:** review the longitudinal and lateral-directional state-space equations.
2. **Why flight modes matter:** connect natural motion to dynamic response, flying qualities, pilot workload, and control design.
3. **Stability concepts:** distinguish stable, neutral, and unstable equilibria, and static stability from dynamic stability.
4. **Analysis tools:** relate SISO transfer-function poles and zeros to MIMO eigenvalues, eigenvectors, controllability, and observability.
5. **Modal decomposition:** use matrix diagonalization and mode superposition to interpret real and oscillatory responses.

Review of Linearized Aircraft Model

$$\mathbf{A}_{long} =$$

$$\begin{bmatrix} \frac{\frac{X_u}{m}}{m-Z_{\dot{w}}} & \frac{\frac{X_w}{m}}{m-Z_{\dot{w}}} & \frac{\frac{X_q}{m} - U_e \sin \theta_e}{Z_q + m U_e \cos \theta_e} & \frac{-g \cos \theta_e}{m-Z_{\dot{w}}} \\ \frac{1}{I_{yy}} \left(M_u + \frac{Z_u}{m-Z_{\dot{w}}} M_{\dot{w}} \right) & \frac{1}{I_{yy}} \left(M_w + \frac{Z_w}{m-Z_{\dot{w}}} M_{\dot{w}} \right) & \frac{1}{I_{yy}} \left(M_q + \frac{Z_q + m U_e \cos \theta_e}{m-Z_{\dot{w}}} M_{\dot{w}} \right) & -\frac{1}{I_{yy}} \left(\frac{mg \sin \theta_e}{m-Z_{\dot{w}}} M_{\dot{w}} \right) \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (1)$$

$$\mathbf{B}_{long} =$$

$$\begin{bmatrix} \frac{\frac{X_{\delta_e}}{m}}{m-Z_{\dot{w}}} & \frac{\frac{X_{\delta_p}}{m}}{m-Z_{\dot{w}}} \\ \frac{1}{I_{yy}} \left(M_{\delta_e} + \frac{Z_{\delta_e}}{m-Z_{\dot{w}}} M_{\dot{w}} \right) & \frac{1}{I_{yy}} \left(M_{\delta_p} + \frac{Z_{\delta_p}}{m-Z_{\dot{w}}} M_{\dot{w}} \right) \\ 0 & 0 \end{bmatrix}$$

Why Study Flight Modes?

The linearized response is a combination of the aircraft's natural modes:

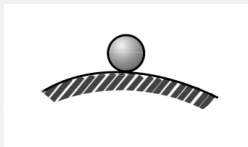
$$\underline{x}(t) = \sum_i c_i \underline{a}_i e^{\lambda_i t}.$$

Each mode has a characteristic time scale, damping, and pattern of motion. Identifying the modes helps to:

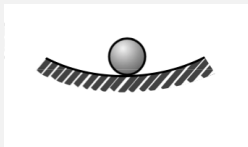
- ▶ interpret the response to disturbances and control inputs;
- ▶ assess stability, handling qualities, and pilot workload;
- ▶ construct useful reduced-order models; and
- ▶ target weakly damped motion through flight-control design.

The phugoid, for example, is a slow exchange between airspeed and altitude. A stable but lightly damped phugoid can persist long enough to affect flight-path control and pilot workload.

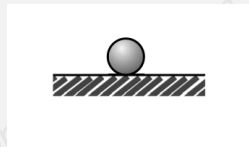
Concepts of Stability



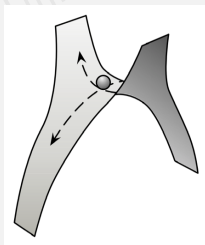
(a) Unstable equilibrium



(b) Stable equilibrium



(c) Neutral equilibrium



(d) Saddle point

Static vs. Dynamic Stability

- ▶ **Static stability:** the *initial tendency* of the aircraft, immediately following a disturbance from equilibrium, to develop a restoring force/moment that pushes it back toward equilibrium. It only concerns the direction of the very first response – not whether the aircraft actually gets back to equilibrium.
- ▶ **Dynamic stability:** concerns the *time history* of the resulting motion after a disturbance – whether it actually damps out and converges back to equilibrium (possibly through oscillations), or grows and diverges over time.

Key relationship: static stability is necessary but **not sufficient** for dynamic stability.

- ▶ statically unstable $\Rightarrow$ dynamically unstable
- ▶ statically stable $\nRightarrow$ dynamically stable (e.g., a divergent oscillation is statically stable but dynamically unstable)

Tools for Stability Analysis and Controllability

- ▶ **Single-input, single-output (SISO) systems:** use transfer functions and pole locations to assess stability and transient response.
- ▶ **Multi-input, multi-output (MIMO) systems:** use state-space models; eigenvalues determine modal stability, while eigenvectors describe the associated patterns of state motion.
- ▶ **Controllability:** determines whether the available control inputs can influence every state-space direction. Controllability permits arbitrary pole placement; stabilizability is sufficient for closed-loop stabilization.

Transfer-Function Analysis

For a linear time-invariant SISO system, the transfer function relates one input to one output under zero initial conditions. A fourth-order model has the form

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}.$$

The roots of the numerator are the *zeros*; the roots of the denominator are the *poles*. Pole locations determine input–output stability and the principal time scales of the response.

Two Approaches to Transfer-Function Analysis

Two complementary forms of analysis are common:

1. **Pole-zero and time-domain analysis:** use pole locations, residues, and step or impulse responses to assess stability, damping, time scales, and modal contributions.
2. **Frequency-domain analysis:** evaluate $G(j\omega)$ using Bode or Nyquist plots to assess resonance, bandwidth, gain and phase, and stability margins.

A transfer function describes only the selected input-output channel; a state-space model exposes the complete multivariable dynamics.

Eigenvalues and Eigenvectors

For a continuous-time MIMO system,

$$\dot{\underline{x}} = \underline{A}\underline{x} + \underline{B}\underline{u}, \quad \underline{y} = \underline{C}\underline{x} + \underline{D}\underline{u},$$

the natural motion is obtained by setting $\underline{u} = \underline{0}$. Each eigenpair satisfies

$$\underline{A}\underline{v}_i = \lambda_i \underline{v}_i, \quad \det(\lambda_i \underline{I} - \underline{A}) = 0.$$

If $\underline{A}$ has a complete set of eigenvectors, then

$$\underline{x}(t) = \sum_i c_i \underline{v}_i e^{\lambda_i t}.$$

- ▶ $\Re(\lambda_i)$ determines modal growth or decay, while $\Im(\lambda_i)$ determines oscillation frequency.
- ▶ $\underline{v}_i$ gives the relative magnitude and phase of the states in that mode; its overall scale is arbitrary.
- ▶ $\underline{B}$ and $\underline{C}$ determine how strongly a mode is controlled and observed.

Eigenanalysis: Toy MIMO Example

Consider the two-input, two-output system

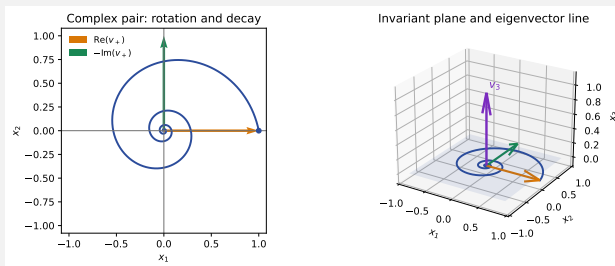
$$\mathbf{A} = \begin{bmatrix} -0.2 & -1 & 0 \\ 1 & -0.2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Python gives the normalized eigenpairs

$$\lambda_{1,2} = -0.2 \pm i, \quad \underline{\mathbf{v}}_{1,2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \mp i & 0 \end{bmatrix}^T, \\ \lambda_3 = -2, \quad \underline{\mathbf{v}}_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T.$$

- ▶ The first pair is a damped oscillation at 1 rad/s with a 5 s decay time constant; x_1 and x_2 have equal magnitude and are 90° out of phase.
- ▶ The third mode is a faster non-oscillatory decay with time constant 0.5 s and involves only x_3 .
- ▶ All eigenvalues lie in the left half-plane, so the unforced system is asymptotically stable.

Eigenanalysis: Toy MIMO Example (Cont'd)



- ▶ An eigenvector specifies a modal direction or shape; its overall scale, sign, and—for a complex eigenvector—phase are arbitrary.
- ▶ For $\lambda_{1,2} = -0.2 \pm i$, $\Re(\underline{v}_+)$ and $-\Im(\underline{v}_+)$ span the invariant x_1 - x_2 plane. The imaginary part of λ produces rotation, while its negative real part contracts the radius, producing the inward spiral.
- ▶ The real eigenvector $\underline{v}_3 = [0 \ 0 \ 1]^T$ defines an invariant line. A state initially aligned with it remains aligned and decays as e^{-2t} .

Solutions of A Linear System

With the control input set to zero, either linearized aircraft model has the form

$$\dot{\underline{x}} = \mathbf{A}\underline{x}.$$

The longitudinal and lateral-directional models considered here each have four states, so $\mathbf{A} \in \mathbb{R}^{4 \times 4}$.

Seek one component of the natural response in the form

$$\underline{x}_i(t) = \underline{v}_i e^{\lambda_i t}.$$

Substitution into the differential equation gives

$$\lambda_i \underline{v}_i = \mathbf{A} \underline{v}_i.$$

Thus, λ_i is an eigenvalue of $\mathbf{A}$ and $\underline{v}_i$ is its corresponding eigenvector.

The eigenvalue determines the time behaviour; the eigenvector describes the relative motion of the states.

Connection Between System Matrix And Transfer Function

For the state-space system

$$\dot{\underline{x}} = \mathbf{A}\underline{x} + \mathbf{B}\underline{u}, \quad \underline{y} = \mathbf{C}\underline{x} + \mathbf{D}\underline{u},$$

the zero-initial-condition transfer matrix is

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}.$$

Since

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\text{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})},$$

the eigenvalues satisfy the characteristic equation

$$\det(s\mathbf{I} - \mathbf{A}) = s^4 + a_3s^3 + a_2s^2 + a_1s + a_0 = 0. \quad (2)$$

The eigenvalues are the internal natural modes and are candidate poles of each transfer-function channel. A pole may be absent from a particular channel because of lack of controllability, lack of observability, or pole-zero cancellation.

Matrix Diagonalization

Assume that $\mathbf{A} \in \mathbb{R}^{n \times n}$ has n linearly independent right eigenvectors. Collect them as the columns of $\mathbf{V}$:

$$\mathbf{A}\mathbf{V} = \mathbf{V}\mathbf{\Lambda}, \quad \mathbf{V} = \begin{bmatrix} \underline{\mathbf{v}}_1 & \cdots & \underline{\mathbf{v}}_n \end{bmatrix}, \quad \mathbf{\Lambda} = \text{diag}(\lambda_1, \dots, \lambda_n).$$

Since $\mathbf{V}$ is invertible,

$$\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \mathbf{\Lambda}, \quad \mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}.$$

Introduce the modal coordinates $\underline{\mathbf{z}} = \mathbf{V}^{-1}\underline{\mathbf{x}}$, or $\underline{\mathbf{x}} = \mathbf{V}\underline{\mathbf{z}}$. The unforced system becomes

$$\dot{\underline{\mathbf{z}}} = \mathbf{\Lambda}\underline{\mathbf{z}}, \quad \dot{z}_i = \lambda_i z_i, \quad z_i(t) = z_i(0)e^{\lambda_i t}.$$

Transforming back to the physical states gives

$$\underline{\mathbf{x}}(t) = \sum_{i=1}^n \underline{\mathbf{v}}_i z_i(0) e^{\lambda_i t}.$$

Diagonalization decouples the dynamics into scalar modal equations; mode superposition reconstructs the physical motion.

Mode Superposition

If $\mathbf{A}$ has four independent eigenvectors, define

$$\mathbf{V} = \begin{bmatrix} \underline{\mathbf{v}}_1 & \underline{\mathbf{v}}_2 & \underline{\mathbf{v}}_3 & \underline{\mathbf{v}}_4 \end{bmatrix}, \quad \mathbf{\Lambda} = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4),$$

so that $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$. Express the initial condition as

$$\underline{\mathbf{x}}(0) = \sum_{i=1}^4 c_i \underline{\mathbf{v}}_i, \quad \underline{\mathbf{c}} = \mathbf{V}^{-1} \underline{\mathbf{x}}(0).$$

The complete unforced response is

$$\underline{\mathbf{x}}(t) = \mathbf{V} e^{\mathbf{\Lambda} t} \underline{\mathbf{c}} = \sum_{i=1}^4 c_i \underline{\mathbf{v}}_i e^{\lambda_i t}.$$

Each term is a *mode*. The coefficient c_i measures how strongly the initial condition excites that mode. Among stable modes, the slowest-decaying mode usually dominates the late-time response.

Real and Complex Modal Responses

For a real-valued matrix $\mathbf{A}$, a real eigenvalue admits a real eigenvector, while every nonreal eigenpair is accompanied by its complex conjugate.

Real eigenvalue. For $\lambda \in \mathbb{R}$, a modal contribution is

$$\underline{\mathbf{x}}_i(t) = c_i \underline{\mathbf{v}}_i e^{\lambda t}.$$

It decays for $\lambda < 0$, remains constant for $\lambda = 0$, and grows for $\lambda > 0$.

Complex-conjugate pair. Let

$$\lambda_{1,2} = \sigma \pm j\omega, \quad \underline{\mathbf{v}}_{1,2} = \underline{\mathbf{v}}_R \pm j\underline{\mathbf{v}}_I, \quad \omega > 0.$$

Combining the conjugate solutions produces the real modal contribution

$$\underline{\mathbf{x}}_{\text{pair}}(t) = e^{\sigma t} \left(\underline{\mathbf{p}} \cos \omega t + \underline{\mathbf{q}} \sin \omega t \right),$$

where $\underline{\mathbf{p}}$ and $\underline{\mathbf{q}}$ are real vectors determined by the eigenvectors and initial condition. Here, σ sets the exponential envelope, ω is the angular frequency, and $T = 2\pi/\omega$ is the period. Asymptotic stability requires $\Re(\lambda_i) < 0$ for every mode.

2. Longitudinal Flight Modes

Phugoid mode (long-period mode):

- ▶ long-period, lightly damped
- ▶ eigenvalues close to the imaginary axis (the dominant pair)
- ▶ significant u and θ motion, with small pitch-rate and angle-of-attack variations
- ▶ approximately constant total energy over one oscillation

Short-period mode:

- ▶ short-period, heavily damped
- ▶ eigenvalues farther from the imaginary axis
- ▶ oscillatory α and q motion with negligible speed variation
- ▶ rapidly damped flight-path response

Study Case: B747-100

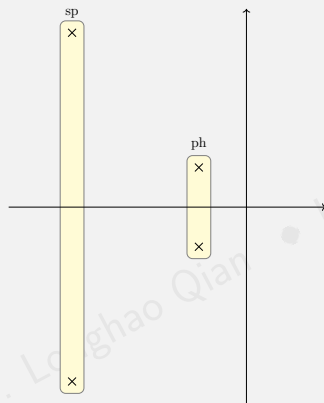
$$\mathbf{A}_{long} = \begin{bmatrix} -0.006866 & 0.01394 & 0 & -9.810 \\ -0.09050 & -0.3149 & 235.893 & 0 \\ 0.0003891 & -0.003362 & -0.4282 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The eigenvalues of $\mathbf{A}_{long}$ are:

$$\lambda_{1,2} = -0.3717 \pm 0.8870i \quad (\text{short-period mode}),$$

$$\lambda_{3,4} = -0.003289 \pm 0.06720i \quad (\text{phugoid mode}).$$

B747-100 Longitudinal Modes



- ▶ All four eigenvalues lie in the left half-plane; the linearized longitudinal dynamics are asymptotically stable.

- ▶ The far-left pair,

$$\lambda_{sp} = -0.3717 \pm 0.8870i,$$

is the rapidly decaying *short-period mode*.

- ▶ The pair near the imaginary axis,

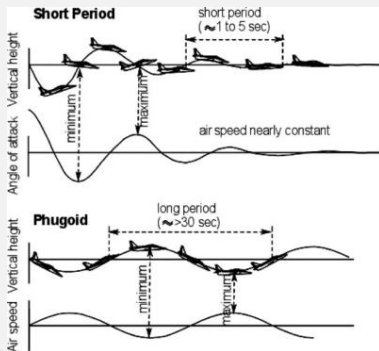
$$\lambda_{ph} = -0.003289 \pm 0.06720i,$$

is the slowly decaying *phugoid mode* and dominates the late-time response.

Physical Picture of the Longitudinal Modes

Short-period mode

- ▶ Rapid oscillation in angle of attack and pitch rate.
- ▶ Airspeed and altitude change little during the first few seconds.



Phugoid mode

- ▶ Slow exchange between kinetic and potential energy.
- ▶ Airspeed is largest near the low point and smallest near the high point of the flight path; angle of attack changes relatively little.

Image source: studfile.net, p. 157 (accessed September 27, 2026).

Phugoid Mode: Definition and Physics

The **phugoid** is a long-period longitudinal oscillation dominated by airspeed, flight-path angle, pitch attitude, and altitude; angle of attack and pitch rate change relatively little.

Its physical mechanism is an exchange between kinetic and gravitational potential energy:

1. A climb converts kinetic energy into potential energy. Airspeed, dynamic pressure, and lift decrease, curving the flight path downward.
2. The ensuing descent converts potential energy back into kinetic energy. Airspeed and lift increase, curving the path upward.
3. The cycle repeats while aerodynamic drag removes energy and provides weak damping.

For the B747-100 case,

$$\lambda_{ph} = -0.003289 \pm 0.06720i \text{ s}^{-1}, \quad \omega_{n,p} = 0.06728 \text{ rad/s}, \quad \zeta_p = 0.0489.$$

Thus, $T_{d,p} \approx 93.5 \text{ s}$, while the amplitude envelope has the much longer time constant $\tau_p \approx 304 \text{ s}$.

Phugoid Mode Approximation

In the phugoid mode, q is small and varies slowly, so set $\dot{q} = 0$ and neglect the q -dependent terms: $M_q = X_q = Z_q = 0$. The rate-derivative terms are also neglected over the long-period motion: $Z_{\dot{w}} = M_{\dot{w}} = 0$. The resulting differential-algebraic model is

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ 0 \\ \dot{\theta} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{X_u}{m} & \frac{X_w}{m} & 0 & -g \\ \frac{Z_u}{m} & \frac{Z_w}{m} & U_e & 0 \\ M_u & M_w & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{A}_p} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} \quad (3)$$

Phugoid Approximation: Characteristic Equation I

The differential-algebraic equation has the form

$$\Pi \dot{x} = Ax, \text{ where } \Pi = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 1 \end{bmatrix}$$

which gives the generalized characteristic equation

$$\det(s\Pi - A) = 0$$

or

$$\begin{vmatrix} s - \frac{X_u}{m} & -\frac{X_w}{m} & 0 & g \\ -\frac{Z_u}{m} & s - \frac{Z_w}{m} & -U_e & 0 \\ -M_u & -M_w & 0 & 0 \\ 0 & 0 & -1 & s \end{vmatrix} = 0 \quad (4)$$

Expansion gives

$$-U_e M_w s^2 + \left[g M_u + \frac{U_e}{m} (X_u M_w - M_u X_w) \right] s + \frac{g}{m} (Z_u M_w - M_u Z_w) = 0 \quad (5)$$

Phugoid Approximation: Characteristic Equation II

Compare this result with the standard second-order characteristic equation

$$s^2 + 2\zeta\omega_n s + \omega_n^2$$

where ω_n is the natural frequency and ζ is the damping ratio. Matching coefficients gives

$$\omega_n^2 = -\frac{g}{mU_e} \left(Z_u - \frac{M_u Z_w}{M_w} \right) \quad (6)$$

$$\zeta = -\frac{1}{2\omega_n} \left[\frac{gM_u}{U_e M_w} + \frac{1}{m} \left(X_u - \frac{M_u}{M_w} X_w \right) \right] \quad (7)$$

Phugoid Approximation: Numerical Comparison

For the full longitudinal model,

$$\lambda_{ph} = \sigma \pm j\omega_d, \quad \omega_n = \sqrt{\sigma^2 + \omega_d^2}, \quad \zeta = -\frac{\sigma}{\omega_n}.$$

Using $\lambda_{ph} = -0.003289 \pm 0.06720i \text{ s}^{-1}$ gives:

Model	$\omega_n^2 \text{ (s}^{-2}\text{)}$	$\omega_n \text{ (rad/s)}$	ζ
Phugoid approximation	0.005068	0.07119	0.0680
Full longitudinal matrix	0.004527	0.06728	0.04888

The approximation overpredicts ω_n by about 5.8% and ζ by about 39%. It captures the long period reasonably well, but the damping is more sensitive to the aerodynamic terms neglected in the reduction.

Short-Period Mode: Definition and Physics

The **short-period mode** is a fast longitudinal oscillation dominated by angle of attack ($\alpha \simeq w/U_e$) and pitch rate q . Airspeed and flight-path angle change relatively little over its short time scale.

Its physical mechanism is a rapid aerodynamic and rotational adjustment:

1. A disturbance changes α , and therefore changes lift and pitching moment almost immediately.
2. Static longitudinal stability ($C_{m_\alpha} < 0$) produces a restoring pitching moment toward the trim angle of attack.
3. Pitch inertia carries the motion past trim, creating an oscillation in α and q .
4. Pitch-rate damping ($C_{m_q} < 0$) removes energy and rapidly reduces the oscillation.

Unlike the phugoid, this mode is not primarily an exchange between kinetic and potential energy. For the B747-100 case,

$$\lambda_{sp} = -0.3717 \pm 0.8870i \text{ s}^{-1}, \quad T_{d,sp} \approx 7.08 \text{ s}, \quad \tau_{sp} \approx 2.69 \text{ s}.$$

Short-Period Mode Approximation

The separation of time scales suggests retaining the fast states w and q while neglecting their weak coupling from the slower speed response. Pitch attitude is not assumed constant: it can be recovered afterward from $\dot{\theta} = q$.

Let $m^* = m - Z_{\dot{w}}$. For nearly level trim ($\cos \theta_e \simeq 1$), the w - q sub-block of the longitudinal model is

$$\begin{bmatrix} \dot{w} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \frac{Z_w}{m^*} & \frac{Z_q + mU_e}{m^*} \\ \frac{1}{I_{yy}} \left(M_w + \frac{M_{\dot{w}}Z_w}{m^*} \right) & \frac{1}{I_{yy}} \left(M_q + \frac{M_{\dot{w}}(Z_q + mU_e)}{m^*} \right) \end{bmatrix} \begin{bmatrix} w \\ q \end{bmatrix}.$$

If $|Z_{\dot{w}}| \ll m$ and $|Z_q| \ll mU_e$, this reduces to

$$\begin{bmatrix} \dot{w} \\ \dot{q} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{Z_w}{m} & U_e \\ \frac{1}{I_{yy}} \left(M_w + \frac{M_{\dot{w}}Z_w}{m} \right) & \frac{1}{I_{yy}} (M_q + M_{\dot{w}}U_e) \end{bmatrix}}_{\mathbf{A}_{sp}} \begin{bmatrix} w \\ q \end{bmatrix}. \quad (8)$$

Short-Period Approximation: Characteristic Equation

For $\mathbf{A}_{sp} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$\det(s\mathbf{I} - \mathbf{A}_{sp}) = s^2 - (a + d)s + (ad - bc).$$

Substituting the entries of Equation (8) gives

$$s^2 - \left[\frac{Z_w}{m} + \frac{M_q + M_{\dot{w}} U_e}{I_{yy}} \right] s + \frac{1}{I_{yy}} \left(\frac{Z_w M_q}{m} - U_e M_w \right) = 0. \quad (9)$$

The $M_{\dot{w}}$ terms cancel from the constant coefficient. Comparing with $s^2 + 2\zeta_{sp}\omega_{n,sp}s + \omega_{n,sp}^2 = 0$ yields

$$\omega_{n,sp}^2 = \frac{1}{I_{yy}} \left(\frac{Z_w M_q}{m} - U_e M_w \right),$$

$$\zeta_{sp} = -\frac{1}{2\omega_{n,sp}} \left[\frac{Z_w}{m} + \frac{M_q + M_{\dot{w}} U_e}{I_{yy}} \right].$$

Thus, stability of this second-order approximation requires a negative trace and a positive determinant for $\mathbf{A}_{sp}$.

Short-Period Approximation: Numerical Comparison

Selecting the w and q rows and columns from the B747-100 matrix gives

$$\mathbf{A}_{sp} = \begin{bmatrix} -0.3149 & 235.893 \\ -0.003362 & -0.4282 \end{bmatrix},$$

and therefore

$$\det(s\mathbf{I} - \mathbf{A}_{sp}) = s^2 + 0.7431s + 0.927912 = 0.$$

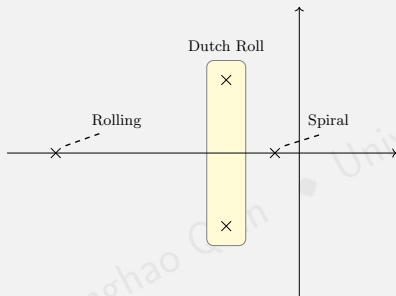
Model	$\lambda_{sp} \text{ (s}^{-1}\text{)}$	$\omega_{n,sp} \text{ (rad/s)}$	ζ_{sp}
Short-period approximation	$-0.37155 \pm 0.88874i$	0.96328	0.38571
Full longitudinal matrix	$-0.37169 \pm 0.88696i$	0.96170	0.38650

The natural-frequency and damping-ratio errors are approximately 0.16% and 0.20%, respectively. The reduction is highly accurate here, but it must be checked against the full matrix for other aircraft and flight conditions.

Lateral-Directional Flight Modes

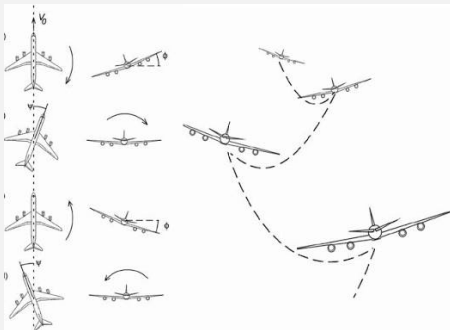
The lateral-directional spectrum normally contains:

- ▶ A far-left real root: the fast, non-oscillatory **roll-subsidence mode**.
- ▶ A real root near the origin: the slow **spiral mode**. Its sign determines whether the motion converges or diverges.
- ▶ A complex pair: the **Dutch-roll mode**, coupling sideslip, yaw, and roll.



All roots shown lie in the left half-plane and are stable. Horizontal distance from the imaginary axis sets the decay rate; vertical distance sets the oscillation frequency.

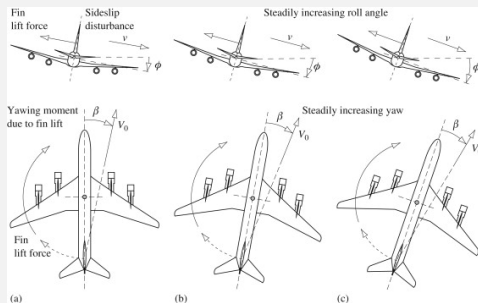
Physical Picture of the Dutch-Roll Mode



- ▶ Dutch roll is an oscillatory coupling of **yaw, sideslip, and roll**.
- ▶ A yaw disturbance creates sideslip. Directional stability turns the nose back into the relative wind, while dihedral effect converts sideslip into roll.
- ▶ Inertia carries the aircraft past equilibrium and reverses the motion. Yaw and roll damping determine how quickly the oscillation decays.

Image source: ResearchGate, "Course of motions of the Dutch roll mode" (accessed September 27, 2026).

Physical Picture of the Spiral Mode



- ▶ The spiral mode is a **slow, non-oscillatory** change in bank and heading.
- ▶ A small bank produces a turn and sideslip. Dihedral effect, directional stability, and gravity determine whether the bank grows or decays.
- ▶ A stable spiral slowly returns toward level flight; an unstable spiral gradually increases bank and turn rate. Its slow development normally permits pilot correction.

Image source: ScienceDirect Topics, "Spiral Mode" (accessed September 27, 2026).

Study Case: B747-100 Lateral-Directional Modes

For $\underline{x}_{latr} = [v \ p \ r \ \phi]^T$ at level trim, the rounded state matrix is

$$\mathbf{A}_{latr} \approx \begin{bmatrix} -0.05577 & 0 & -235.900 & 9.810 \\ -0.01270 & -0.4351 & 0.4143 & 0 \\ 0.003567 & -0.006056 & -0.1458 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Using the unrounded model, the characteristic equation is

$$s^4 + 0.63451s^3 + 0.93747s^2 + 0.50975s + 0.0036607 = 0.$$

Its roots identify three distinct lateral-directional modes:

Mode	Eigenvalue (s^{-1})	Key measure
Spiral	-0.007278	$\tau \approx 137 \text{ s}$
Roll subsidence	-0.56138	$\tau \approx 1.78 \text{ s}$
Dutch roll	$-0.03293 \pm 0.94597i$	$\omega_n \approx 0.9465 \text{ rad/s}, \zeta \approx 0.0348$

All modes are stable, but the spiral mode is very slow and the Dutch roll is lightly damped.

Physical Interpretation of the Lateral Modes

- ▶ **Roll subsidence:** A roll-rate disturbance changes the wing angles of attack. The resulting roll-damping moment ($C_{l_p} < 0$) opposes p , producing a fast, nearly non-oscillatory decay.
- ▶ **Spiral mode:** A small bank angle initiates a turn and sideslip. Directional stability, dihedral effect, and gravity then compete. Their near balance produces a very slow convergence or divergence, so the spiral eigenvalue lies close to the origin.
- ▶ **Dutch roll:** A sideslip generates a restoring yawing moment ($C_{n_\beta} > 0$). Yaw inertia carries the aircraft past equilibrium, while dihedral effect couples the sideslip into roll. Yaw and roll damping reduce the resulting oscillation.

The mode shapes depend on state units and eigenvector normalization; their physical interpretation should therefore emphasize relative participation and phase rather than raw eigenvector magnitudes alone.

Spiral and Roll-Subsidence Approximation

To isolate the two real modes, treat the lateral side-force equation as quasi-steady ($\dot{v} \simeq 0$), neglect its smaller aerodynamic derivatives, and retain the dominant kinematic coupling $\mathcal{Y}_r \simeq -U_e$. For level trim, $U_e r \simeq g\phi$, and

$$\det \begin{bmatrix} 0 & 0 & U_e & -g \\ -\mathcal{L}_v & s - \mathcal{L}_p & -\mathcal{L}_r & 0 \\ -\mathcal{N}_v & -\mathcal{N}_p & s - \mathcal{N}_r & 0 \\ 0 & -1 & 0 & s \end{bmatrix} = 0.$$

Expanding the determinant gives the verified quadratic

$$U_e \mathcal{N}_v s^2 + [U_e (\mathcal{L}_v \mathcal{N}_p - \mathcal{L}_p \mathcal{N}_v) - g \mathcal{L}_v] s + g (\mathcal{L}_v \mathcal{N}_r - \mathcal{L}_r \mathcal{N}_v) = 0.$$

Using the displayed rounded B747-100 matrix,

Mode	Approximation (s^{-1})	Full matrix (s^{-1})
Spiral	-0.007295	-0.007252
Roll subsidence	-0.59743	-0.56322

The approximation captures the spiral root closely and slightly overpredicts the roll-subsidence decay rate.

Dutch Roll Mode Approximation

Retain the sideslip velocity v and yaw rate r , and neglect their coupling through p and ϕ . With $\mathcal{Y}_r \simeq -U_e$, the reduced model is

$$\begin{bmatrix} \dot{v} \\ \dot{r} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathcal{Y}_v & -U_e \\ \mathcal{N}_v & \mathcal{N}_r \end{bmatrix}}_{\mathbf{A}_{dr}} \begin{bmatrix} v \\ r \end{bmatrix}.$$

Hence,

$$\det(s\mathbf{I} - \mathbf{A}_{dr}) = s^2 - (\mathcal{Y}_v + \mathcal{N}_r)s + (\mathcal{Y}_v\mathcal{N}_r + U_e\mathcal{N}_v) = 0.$$

Therefore,

$$\omega_{n,dr}^2 = \mathcal{Y}_v\mathcal{N}_r + U_e\mathcal{N}_v, \quad \zeta_{dr} = -\frac{\mathcal{Y}_v + \mathcal{N}_r}{2\omega_{n,dr}}.$$

For the displayed rounded B747-100 matrix,

$$s^2 + 0.20157s + 0.849587 = 0 \implies s = -0.10079 \pm 0.91620i,$$

compared with $s = -0.03310 \pm 0.94702i$ from the full matrix. The reduced model captures the frequency reasonably well but overpredicts the damping because roll coupling has been neglected.

4. Longitudinal Response

For the linearized system

$$\dot{\underline{x}} = \underline{A}\underline{x} + \underline{B}\underline{u}, \quad \underline{y} = \underline{C}\underline{x} + \underline{D}\underline{u},$$

the **response** is the state and output time history produced by an initial condition and applied inputs:

$$\underline{x}(t) = \underbrace{e^{\underline{A}t}\underline{x}(0)}_{\text{natural (free) response}} + \underbrace{\int_0^t e^{\underline{A}(t-\tau)} \underline{B}\underline{u}(\tau) d\tau}_{\text{forced response}}.$$

- ▶ **Modal analysis** describes how the aircraft moves naturally when $\underline{u} = 0$: stability, damping, frequency, and mode shape.
- ▶ **Response analysis** predicts what is observed after an elevator, throttle, gust, or command input.
- ▶ It quantifies transient and steady-state performance: rise time, overshoot, settling time, tracking accuracy, and control effort.

Open-loop inputs excite the natural modes; feedback changes the closed-loop system matrix and therefore changes both its modes and response.

State-Space Response

The linearized longitudinal equations are

$$\dot{\underline{x}} = \mathbf{A}_{long} \underline{x} + \mathbf{B}_{long} \underline{u}$$

where $\underline{x} = [u \ w \ q \ \theta]^T$ and $\underline{u} = [\delta_e \ \delta_p]^T$. With zero initial conditions, the Laplace transform gives the transfer matrix

$$\underline{x}(s) = (s\mathbf{1} - \mathbf{A}_{long})^{-1} \mathbf{B}_{long} \underline{u}(s)$$

Longitudinal Transfer Matrix

Specifically,

$$\begin{bmatrix} u(s) \\ w(s) \\ q(s) \\ \theta(s) \end{bmatrix} = \begin{bmatrix} G_{u\delta_e} & G_{u\delta_p} \\ G_{w\delta_e} & G_{w\delta_p} \\ G_{q\delta_e} & G_{q\delta_p} \\ G_{\theta\delta_e} & G_{\theta\delta_p} \end{bmatrix} \begin{bmatrix} \delta_e(s) \\ \delta_p(s) \end{bmatrix}$$

$$= \frac{1}{D(s)} \begin{bmatrix} N_{u\delta_e}(s) & N_{u\delta_p}(s) \\ N_{w\delta_e}(s) & N_{w\delta_p}(s) \\ N_{q\delta_e}(s) & N_{q\delta_p}(s) \\ N_{\theta\delta_e}(s) & N_{\theta\delta_p}(s) \end{bmatrix} \begin{bmatrix} \delta_e(s) \\ \delta_p(s) \end{bmatrix} \quad (10)$$

where $D(s) = \det(s\mathbf{1} - \mathbf{A}_{long})$ is the characteristic polynomial of the longitudinal modes.

B747-100 Longitudinal Transfer Functions

For the B747-100 model, the transfer functions are

$$G_{u\delta_e}(s) = \frac{-5.726 \times 10^{-5}s^3 - 0.07684s^2 + 7.511s + 3.392}{s^4 + 0.7499s^3 + 0.9341s^2 + 0.009449s + 0.004187}$$

$$G_{w\delta_e}(s) = \frac{-5.508s^3 - 275.3s^2 - 1.89s - 1.048}{s^4 + 0.7499s^3 + 0.9341s^2 + 0.009449s + 0.004187}$$

$$G_{q\delta_e}(s) = \frac{-1.157s^3 - 0.3537s^2 - 0.003864s}{s^4 + 0.7499s^3 + 0.9341s^2 + 0.009449s + 0.004187}$$

$$G_{\theta\delta_e}(s) = \frac{-1.157s^2 - 0.3537s - 0.003864}{s^4 + 0.7499s^3 + 0.9341s^2 + 0.009449s + 0.004187}$$

$$G_{u\delta_p}(s) = \frac{2.943s^3 + 2.187s^2 + 2.73s}{s^4 + 0.7499s^3 + 0.9341s^2 + 0.009449s + 0.004187}$$

$$G_{w\delta_p}(s) = \frac{-0.2664s^2 + 0.1561s}{s^4 + 0.7499s^3 + 0.9341s^2 + 0.009449s + 0.004187}$$

$$G_{q\delta_p}(s) = \frac{0.001145s^2 + 0.001256s}{s^4 + 0.7499s^3 + 0.9341s^2 + 0.009449s + 0.004187}$$

$$G_{\theta\delta_p}(s) = \frac{0.001145s + 0.001256}{s^4 + 0.7499s^3 + 0.9341s^2 + 0.009449s + 0.004187}$$

From a MIMO Model to a Channel Response

Question 1. How can one evaluate the dynamic response and potential feedback-control requirements of a MIMO aircraft model?

1. Select an input–output channel from the transfer matrix.
2. Examine its poles, zeros, and steady-state gain.
3. Apply a representative command or disturbance and compute the time response.
4. Compare the response with performance requirements such as damping, settling time, overshoot, and steady-state error.

B747-100 Elevator-to-Pitch Channel

The transfer function $G_{\theta\delta_e}$ has poles and zeros

$$p_{1,2} = -0.3717 \pm 0.8869i \quad \text{short period}$$

$$p_{3,4} = -0.0033 \pm 0.0672i \quad \text{phugoid}$$

$$z_1 = -0.2944$$

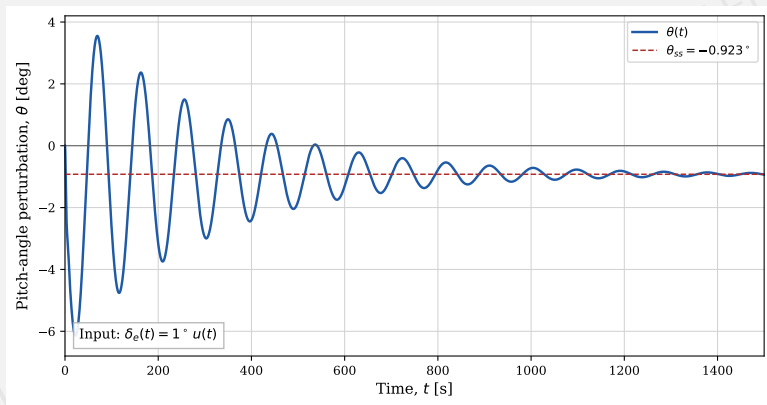
$$z_2 = -0.0113$$

All poles lie in the left half-plane, but the lightly damped phugoid pair lies close to the imaginary axis and governs the long settling time. The DC gain is

$$G_{\theta\delta_e}(0) = \frac{-0.003864}{0.004187} = -0.923.$$

For a positive 1° elevator step, the final-value theorem therefore predicts $\theta_{ss} = -0.923^\circ$ under the stated sign convention.

Pitch Response to a 1° Elevator Step



The fast short-period transient decays first; the lightly damped phugoid then dominates the long-time response.

Control-Performance Questions

Question 2. Which features of this open-loop response matter for feedback-control design?

- ▶ stability and modal damping;
- ▶ peak excursion and oscillation;
- ▶ settling time;
- ▶ steady-state gain, sign, and tracking error; and
- ▶ required control effort and actuator limits.

Open-Loop Response Assessment

The response is stable, but its control performance is poor:

- ▶ The lightly damped phugoid produces a large, slowly decaying oscillation.
- ▶ The pitch angle ranges from approximately -6.14° to $+3.55^\circ$, a peak-to-peak excursion of 9.70° .
- ▶ The 2% settling time is approximately 1710 s (28.5 min).
- ▶ The negative DC gain gives $\theta_{ss} = \lim_{s \rightarrow 0} s\Theta(s) = -0.923^\circ$ for a positive 1° elevator step under the stated sign convention.

Feedback control must therefore increase modal damping, reduce the transient excursion and settling time, and provide the required steady-state response without exceeding actuator limits.

Phugoid Elevator-to-Pitch Approximation

$$G_{\theta\delta_e}^{ph} = \frac{c_2 s^2 + c_1 s + c_0}{A s^2 + B s + C} \quad (11)$$

where

$$A = -U_e M_w$$

$$B = g M_u + \frac{U_e}{m} (X_u M_w - M_u X_w) \approx \frac{U_e}{m} X_u M_w$$

$$C = \frac{g}{m} (Z_u M_w - M_u Z_w) \approx \frac{g}{m} Z_u M_w$$

$$c_2 = M_{\delta_e}$$

$$c_1 = \frac{1}{m} (M_u X_{\delta_e} + M_w Z_{\delta_e}) - \frac{M_{\delta_e}}{m} (X_u + Z_w) \approx -\frac{M_{\delta_e}}{m} (X_u + Z_w)$$

$$c_0 = \frac{M_{\delta_e}}{m^2} (X_u Z_w - X_w Z_u) + \frac{Z_{\delta_e}}{m^2} (M_u X_w - X_u M_w) + \frac{X_{\delta_e}}{m^2} (M_w Z_u - M_u Z_w)$$

$$\approx M_{\delta_e} \frac{X_u}{m} \frac{Z_w}{m}$$

Phugoid Approximation: B747-100 Comparison

The complete phugoid reduction gives

$$G_{\theta\delta_e}^{ph}(s) = \frac{-1.4114s^2 - 0.42802s - 0.0046745}{s^2 + 0.0096834s + 0.0050643},$$

whereas retaining only the leading coefficient terms gives

$$G_{\theta\delta_e}^{ph, lead}(s) = \frac{-1.4114s^2 - 0.45121s - 0.0030316}{s^2 + 0.0068662s + 0.0037384}.$$

Model	Poles (s^{-1})	Zeros (s^{-1})	$G(0)$
Full model	$-0.003289 \pm 0.067202i$	$-0.29442, -0.011344$	-0.92303
Phugoid reduction	$-0.004842 \pm 0.070999i$	$-0.29191, -0.011346$	-0.92303
Leading terms	$-0.003433 \pm 0.061046i$	$-0.31282, -0.006866$	-0.81092

The complete reduction preserves the full-model DC gain. For a positive 1° elevator step, the leading-term model predicts $\theta_{ss} = -0.811^\circ$, compared with -0.923° for the complete reduction and full model.

B747-100 Elevator-to-Speed Channel

The transfer function $G_{u\delta_e}(s)$ has the following poles and zeros:

$$p_{1,2} = -0.3717 \pm 0.8869i \quad \text{short period}$$

$$p_{3,4} = -0.0033 \pm 0.0672i \quad \text{phugoid}$$

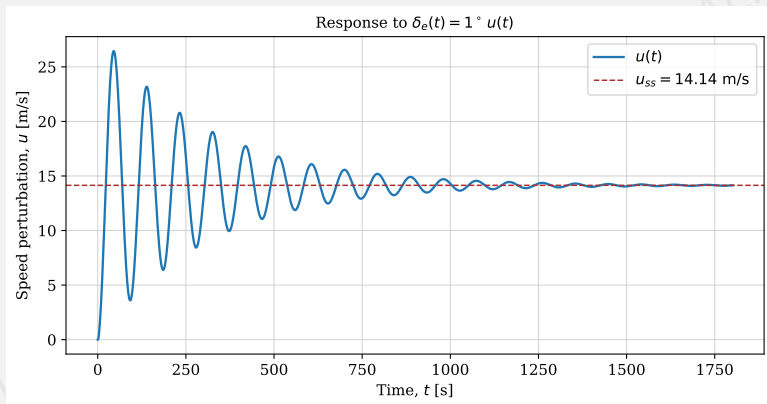
$$z_1 = -1433.3$$

$$z_2 = -0.4496$$

$$z_3 = +91.9$$

The right-half-plane zero makes this channel nonminimum phase, although its high frequency limits its influence on the long-period response. The DC gain is 810.33 (m/s)/rad, so a positive 1° elevator step gives $u_{ss} = 14.14$ m/s.

Speed Response to a 1° Elevator Step



The long-time oscillation is governed by the lightly damped phugoid pair.

Phugoid Elevator-to-Speed Approximation

The reduced transfer function is

$$G_{u\delta_e}^{ph} = \frac{a_1 s + a_0}{As^2 + Bs + C} \quad (12)$$

where

$$\begin{aligned} a_1 &= \frac{U_e}{m} (M_{\delta_e} X_w - X_{\delta_e} M_w) - g M_{\delta_e} \approx -g M_{\delta_e} \\ a_0 &= \frac{g}{m} (M_{\delta_e} Z_w - Z_{\delta_e} M_w) \approx g M_{\delta_e} \frac{Z_w}{m} \end{aligned}$$

Numerically,

$$G_{u\delta_e}^{ph}(s) = \frac{9.2032s + 4.1038}{s^2 + 0.0096834s + 0.0050643},$$

while the further leading-term approximation is

$$G_{u\delta_e}^{ph, lead}(s) = \frac{13.8458s + 4.3313}{s^2 + 0.0068662s + 0.0037384}.$$

For a positive 1° elevator step, the complete reduction predicts $u_{ss} = 14.14$ m/s; the leading-term approximation predicts 20.22 m/s.

B747-100 Throttle-to-Speed Channel

The throttle initially changes airspeed directly. For $G_{u\delta_p}(s)$, the poles and zeros are

$$p_{1,2} = -0.3717 \pm 0.8869i \quad \text{short period}$$

$$p_{3,4} = -0.0033 \pm 0.0672i \quad \text{phugoid}$$

$$z_{1,2} = -0.3715 \pm 0.8887i$$

$$z_3 = 0$$

Interpretation of the Throttle-to-Speed Channel

The complex zeros nearly cancel the short-period poles, so the throttle response is dominated by the phugoid. The zero at the origin implies

$$G_{u\delta_p}(0) = 0 :$$

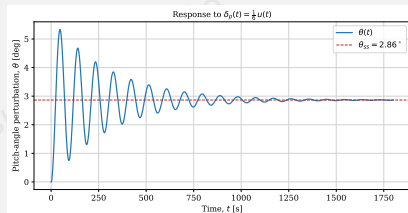
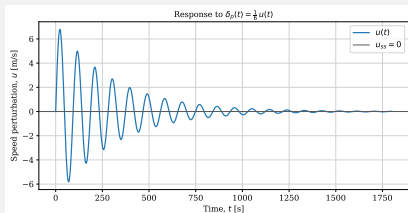
after a constant throttle change, the speed perturbation returns to zero in this unconstrained linear model while the flight-path angle changes.

The assumed derivatives are

$$X_{\delta_p} = 0.3mg, \quad Z_{\delta_p} = M_{\delta_p} = 0.$$

The example uses $\delta_p = 1/6$, corresponding to $\Delta T = 0.05mg$.

Responses to a $\delta_p = 1/6$ Throttle Step



The speed response is transient, whereas the pitch and flight-path angles approach nonzero steady values.

Phugoid Throttle-to-Pitch Approximation

With $Z_{\delta_p} = M_{\delta_p} = 0$, the phugoid approximation is

$$G_{\theta\delta_p}^{ph}(s) = \frac{e_1 s + e_0}{As^2 + Bs + C}, \quad (13)$$

where

$$e_1 = \frac{M_u X_{\delta_p}}{m}, \quad e_0 = \frac{X_{\delta_p}}{m^2} (M_w Z_u - M_u Z_w).$$

If the M_u terms are small,

$$e_1 \approx 0, \quad e_0 \approx \frac{X_{\delta_p} M_w Z_u}{m^2}, \quad G_{\theta\delta_p}^{ph}(0) \approx \frac{X_{\delta_p}}{mg} = 0.3.$$

Consequently, for $\delta_p = 1/6$,

$$\theta_{ss} \approx \gamma_{ss} = 0.3 \left(\frac{1}{6} \right) = 0.05 \text{ rad} = 2.86^\circ.$$

B747-100 Elevator-to-Pitch-Rate Channel

The pitch-rate response is determined by $G_{q\delta_e}(s)$. Its poles and zeros are

$$p_{1,2} = -0.3717 \pm 0.8869i \quad \text{short period}$$

$$p_{3,4} = -0.0033 \pm 0.0672i \quad \text{phugoid}$$

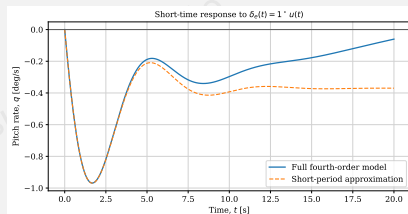
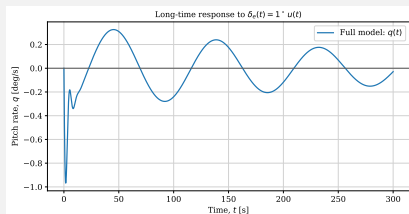
$$z_1 = -0.2944$$

$$z_2 = -0.0113$$

$$z_3 = 0$$

Since $q = \dot{\theta}$, zero initial conditions imply $G_{q\delta_e}(s) = sG_{\theta\delta_e}(s)$. The zero at the origin gives $G_{q\delta_e}(0) = 0$, so the pitch rate returns to zero after a constant elevator step even though pitch angle approaches a nonzero value.

Pitch-Rate Response to a 1° Elevator Step



The short-period approximation reproduces the initial transient; the full model is required for the residual phugoid motion.

Short-Period Model with Elevator Input

After neglecting $Z_{\dot{w}}$ relative to m and Z_q relative to mU_e , the controlled short-period model is

$$\begin{bmatrix} \dot{w} \\ \dot{q} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{Z_w}{m} & U_e \\ \frac{M_w + M_{\dot{w}}Z_w/m}{I_{yy}} & \frac{M_q + M_{\dot{w}}U_e}{I_{yy}} \end{bmatrix}}_{\mathbf{A}_{sp}} \begin{bmatrix} w \\ q \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{Z_{\delta_e}}{m} \\ \frac{M_{\delta_e} + M_{\dot{w}}Z_{\delta_e}/m}{I_{yy}} \end{bmatrix}}_{\mathbf{b}_{sp,e}} \delta_e.$$

The $M_{\dot{w}}Z_{\delta_e}/m$ contribution must be retained in the input matrix whenever $M_{\dot{w}}$ is retained in $\mathbf{A}_{sp}$.

Short-Period Elevator-to-Pitch-Rate Transfer Function

For

$$G_{q\delta_e}^{sp}(s) = \frac{d_1 s + d_0}{s^2 + Ds + E},$$

the coefficients are

$$D = - \left[\frac{Z_w}{m} + \frac{M_q + M_{\dot{w}} U_e}{I_{yy}} \right],$$

$$E = - \frac{U_e M_w}{I_{yy}} + \frac{Z_w M_q}{m I_{yy}},$$

$$d_1 = \frac{M_{\delta_e} + M_{\dot{w}} Z_{\delta_e} / m}{I_{yy}},$$

$$d_0 = \frac{Z_{\delta_e} M_w - Z_w M_{\delta_e}}{m I_{yy}}.$$

The $M_{\dot{w}}$ terms cancel from E and d_0 . The further leading-term reduction neglects $M_{\dot{w}}$ in D , d_1 , $Z_w M_q / (m I_{yy})$ in E , and $Z_{\delta_e} M_w / (m I_{yy})$ in d_0 .

Short-Period Transfer Function: B747-100 Comparison

The complete short-period reduction gives

$$G_{q\delta_e}^{sp}(s) = \frac{-1.15695s - 0.34352}{s^2 + 0.74100s + 0.92715},$$

and the further leading-term approximation gives

$$G_{q\delta_e}^{sp, lead}(s) = \frac{-1.15902s - 0.36257}{s^2 + 0.65158s + 0.82118}.$$

Model	Poles (s^{-1})	Zero (s^{-1})
Full model (short pair)	$-0.37168 \pm 0.88692i$	—
Short-period reduction	$-0.37050 \pm 0.88875i$	-0.29692
Leading terms	$-0.32579 \pm 0.84560i$	-0.31282

The complete reduction closely matches the full short-period poles; most of the error appears only after the additional leading-term omissions.

5. Lateral Responses

The lateral responses are derived from the lateral state equations.
Based on the linearize lateral equations,

$$\dot{\underline{x}} = \mathbf{A}_{latr} \underline{x} + \mathbf{B}_{latr} \underline{u}$$

where $\underline{x} = [v \ p \ r \ \phi]^T$, $\underline{u} = [\delta_a \ \delta_r]^T$. The transfer functions (matrices) are obtained by taking Laplace transform, as

$$\underline{x}(s) = (s\mathbf{1} - \mathbf{A}_{latr})^{-1} \mathbf{B}_{latr} \underline{u}(s) \quad (14)$$

or specifically,

$$\begin{aligned}
 \begin{bmatrix} v(s) \\ p(s) \\ r(s) \\ \phi(s) \end{bmatrix} &= \begin{bmatrix} G_{v\delta_a} & G_{v\delta_r} \\ G_{p\delta_a} & G_{p\delta_r} \\ G_{r\delta_a} & G_{r\delta_r} \\ G_{\phi\delta_a} & G_{\phi\delta_r} \end{bmatrix} \begin{bmatrix} \delta_a(s) \\ \delta_r(s) \end{bmatrix} \\
 &= \frac{1}{D(s)} \begin{bmatrix} N_{v\delta_a}(s) & N_{v\delta_r}(s) \\ N_{p\delta_a}(s) & N_{p\delta_r}(s) \\ N_{r\delta_a}(s) & N_{r\delta_r}(s) \\ N_{\phi\delta_a}(s) & N_{\phi\delta_r}(s) \end{bmatrix} \begin{bmatrix} \delta_a(s) \\ \delta_r(s) \end{bmatrix} \quad (15)
 \end{aligned}$$

where $D(s)$ is the exactly the characteristic equation of lateral modes. We will pick the rudder step response with [B747-100] case to illustrate the lateral responses.

δ_r Step Responses

In [B747-100] case, the transfer functions corresponding to the rudder input is shown here.

$$\begin{aligned} G(v\delta_r) &= \frac{1.7188s^3 + 115.6179s^2 + 51.2499s - 1.8105}{s^4 + 0.63646s^3 + 0.9393s^2 + 0.51216s + 0.0036776} \\ G(p\delta_r) &= \frac{0.11463s^3 - 0.2s^2 - 1.3703s}{s^4 + 0.63646s^3 + 0.9393s^2 + 0.51216s + 0.0036776} \\ G(r\delta_r) &= \frac{-0.48588s^3 - 0.23297s^2 - 0.0090255s - 0.056529}{s^4 + 0.63646s^3 + 0.9393s^2 + 0.51216s + 0.0036776} \\ G(\phi\delta_r) &= \frac{0.11463s^2 - 0.2s - 1.3703}{s^4 + 0.63646s^3 + 0.9393s^2 + 0.51216s + 0.0036776} \end{aligned}$$

For example, when we give the rudder a positive 1 deg step input, the rudder is deflected to the left, based on the static analysis, we shall expect the aircraft yaws to the left with the negative yawing rate (r).

Rudder-to-Yaw-Rate Step Response

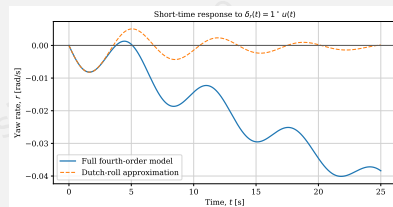
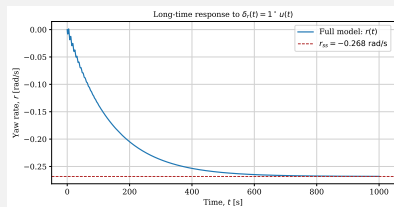


Figure 1: Step Response: $\delta_r \rightarrow r$

Steady Yaw Rate and Dutch-Roll Approximation

For a positive 1° rudder step, the B747-100 steady yaw rate is

$$\begin{aligned} r_{ss} &= \lim_{t \rightarrow \infty} r(t) = \lim_{s \rightarrow 0} sR(s) \\ &= -\frac{0.056529}{0.0036776} \left(\frac{\pi}{180} \right) = -0.268 \text{ rad/s.} \end{aligned}$$

The short-term behavior can be captured using the Dutch-roll approximation:

$$\begin{bmatrix} \dot{v} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \mathcal{Y}_v & -U_e \\ \mathcal{N}_v & \mathcal{N}_r \end{bmatrix} \begin{bmatrix} v \\ r \end{bmatrix} + \begin{bmatrix} \mathcal{Y}_{\delta_a} & \mathcal{Y}_{\delta_r} \\ \mathcal{N}_{\delta_a} & \mathcal{N}_{\delta_r} \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix}$$

Dutch-Roll Rudder-to-Yaw-Rate Approximation

The reduced model gives the approximate transfer function

$$G_{r\delta_r}^{dr} = \frac{f_1 s + f_0}{s^2 + F s + H} \quad (16)$$

where

$$F = -(\mathcal{Y}_v + \mathcal{N}_r)$$

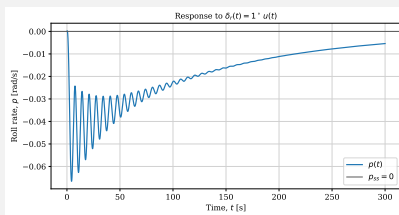
$$H = \mathcal{Y}_v \mathcal{N}_r + U_e \mathcal{N}_v$$

$$f_1 = \mathcal{N}_{\delta_r}$$

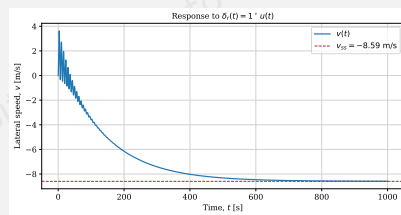
$$f_0 = \mathcal{N}_v \mathcal{Y}_{\delta_r} - \mathcal{N}_{\delta_r} \mathcal{Y}_v$$

It captures the short-term Dutch-roll response but omits the long-term spiral effect, as shown in Figure 1.

Other Rudder Step Responses



Rudder-to-roll-rate response, $\delta_r \rightarrow p$.



Rudder-to-lateral-speed response, $\delta_r \rightarrow v$.

Summary of Open-loop Modes and Response

Longitudinal modes

- ▶ **Short period:** fast oscillation dominated by angle of attack and pitch rate; it governs the initial pitch response and handling qualities.
- ▶ **Phugoid:** slow, lightly damped exchange between kinetic and potential energy; airspeed, altitude, and flight-path angle may oscillate for many cycles.

Lateral-directional modes

- ▶ **Roll subsidence:** fast, non-oscillatory decay of roll rate.
- ▶ **Dutch roll:** coupled yaw–sideslip–roll oscillation; weak damping degrades handling and ride quality.
- ▶ **Spiral:** very slow bank and heading motion; it may converge weakly or diverge if its real pole moves into the RHP.

From modes to response. Open-loop poles determine stability and time scales, and eigenvectors describe state participation. The observed response also depends on the selected input/output channel and its zeros, and is generally a superposition of fast and slow modal contributions.

Why control is necessary

Open-loop stability alone does not guarantee acceptable behavior. Feedback is needed to stabilize unstable modes, increase damping, shape handling qualities, track commands, and reject disturbances and model uncertainty.

Summary

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- 2 Longitudinal Flight Modes
- 3 Lateral Flight Modes
- 4 Longitudinal Response
- 5 Lateral Responses