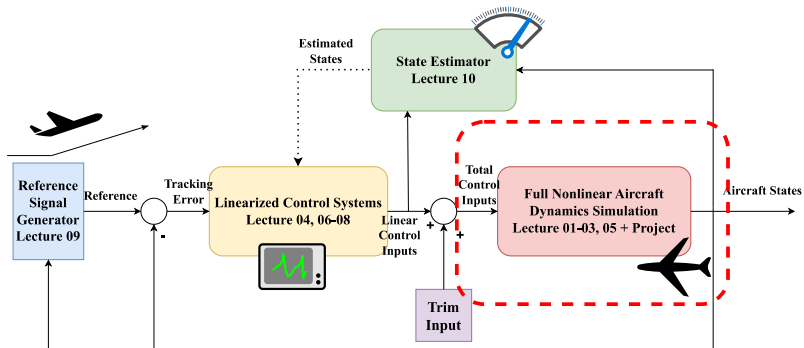


Lecture 02

Flight Dynamics and Linearization

September 23, 2026

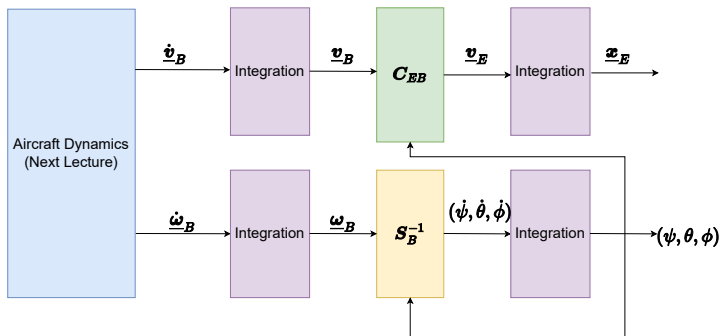
Acknowledgement: These slides were developed with reference to course materials by Professor Hugh H. T. Liu.



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- 1 Rigid-Body Dynamics
- 2 Applied Flight Forces and Moments
- 3 Flight Dynamics: Linearization

Review: Aircraft Motion



Flight Dynamics

- ▶ simulation: applied forces and moments
- ▶ governing principle: Newton's Second Law
- ▶ rigid-body assumption

1. Rigid-Body Dynamics

A rigid body is composed of numerous little mass elements: $\Delta m_i, i = 1, \dots, N$. This will allow us to treat the rigid body as a large collection of particles:

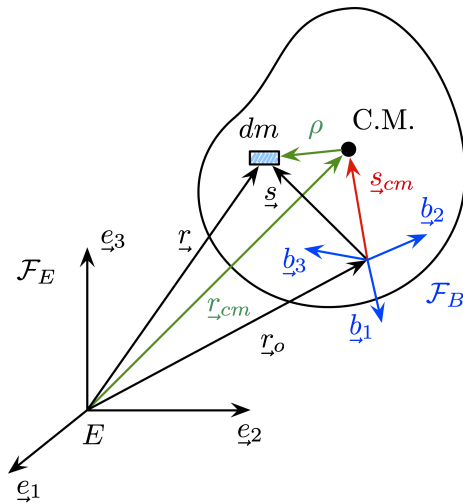
$$m = \lim_{N \rightarrow \infty} \sum_{i=1}^N \Delta m_i = \int_{\mathcal{B}} dm$$

The position of the *centre of mass* of the rigid body $\mathcal{B}$ is given by

$$\underline{\mathbf{r}}_{cm} \triangleq \frac{1}{m} \int_{\mathcal{B}} \underline{\mathbf{r}} dm \quad (1)$$

The inertial coordinates have the following properties:

$$\underline{\mathbf{r}}_{cm,E} = \frac{1}{m} \int_{\mathcal{B}} (\underline{\mathbf{r}}_{cm,E} + \underline{\mathbf{s}}_E) dm \rightarrow \int_{\mathcal{B}} \underline{\mathbf{s}}_E dm = \underline{\mathbf{0}} \quad (2)$$



Mathematical Definition of Rigid-body And Center of Mass

The coordinate of $\underline{s}$ in $\mathcal{F}_B$ is $\underline{s}_B$. The inertial coordinate of the infinitesimal mass element $\underline{r}_E$ is

$$\underline{r}_E = \underline{r}_{cm,E} + \mathbf{C}_{EB}\underline{s}_B$$

For a rigid body, the time derivative of $\underline{s}_B$ is zero:

$$\dot{\underline{s}}_B = \underline{0}$$

Hence, the linear velocity of this mass element is:

$$\underline{v}_E = \dot{\underline{r}}_E = \dot{\underline{r}}_{cm,E} + \mathbf{C}_{EB}\underline{\omega}_B^\times \underline{s}_B = \underline{v}_{cm,E} + \mathbf{C}_{EB}\underline{\omega}_B^\times \underline{s}_B$$

where $\underline{v}_{cm,E}$ is the inertial velocity of the center of mass of the rigid-body.

Linear Momentum in Inertial Frame

The total momentum in inertial frame $\underline{\mathbf{p}}_E$ is obtained by volume integration

$$\begin{aligned}
 \underline{\mathbf{p}}_E &= \lim_{N \rightarrow \infty, \Delta m_i \rightarrow 0} \sum_{i=1}^N \Delta m_i \underline{\mathbf{v}}_E = \int_B \underline{\mathbf{v}}_E dm \\
 &= \int_B \underline{\mathbf{v}}_{cm,E} + \mathbf{C}_{EB} \underline{\boldsymbol{\omega}}_B^\times \underline{\mathbf{s}}_B dm = m \underline{\mathbf{v}}_{cm,E} + \mathbf{C}_{EB} \underline{\boldsymbol{\omega}}_B^\times \int_B \underline{\mathbf{s}}_B dm \\
 &= m \underline{\mathbf{v}}_{cm,E}
 \end{aligned}$$

Translational Dynamics

Applying Newton's Second Law, the time-derivative of linear momentum equals the total net applied force.

$$\begin{aligned}\underline{\mathbf{f}}_E &= \underline{\dot{\mathbf{p}}}_E = m \underline{\dot{\mathbf{v}}}_{cm,E} \\ &= m \frac{d}{dt} \mathbf{C}_{EB} \underline{\mathbf{v}}_{cm,B} = m \mathbf{C}_{EB} (\underline{\dot{\mathbf{v}}}_{cm,B} + \underline{\boldsymbol{\omega}}_B^\times \underline{\mathbf{v}}_{cm,B}) \\ &\rightarrow \underline{\mathbf{f}}_B = m (\underline{\dot{\mathbf{v}}}_{cm,B} + \underline{\boldsymbol{\omega}}_B^\times \underline{\mathbf{v}}_{cm,B})\end{aligned}\quad (3)$$

Angular Momentum about CM

The angular momentum (about Point CM) is expressed as

$$\underline{h}_{cm} = \int_B \underline{s} \times d\underline{p} \quad (4)$$

In coordinate form,

$$\underline{h}_{cm,E} = \int_B \underline{s}_E^\times d\underline{p}_E = \int_B \mathbf{C}_{EB} \underline{s}_B^\times \mathbf{C}_{BE} dm (\underline{v}_{cm,E} + \mathbf{C}_{EB} \underline{\omega}_B^\times \underline{s}_B)$$

Note that for cross product matrix, we have the following identity:

$$\underline{s}_E^\times = (\mathbf{C}_{EB} \underline{s}_B)^\times = \mathbf{C}_{EB} \underline{s}_B^\times \mathbf{C}_{BE}$$

$$\begin{aligned}
 \underline{h}_{cm,E} &= \mathbf{C}_{EB} \int_B \underline{s}_B^\times \mathbf{C}_{BE} (\underline{v}_{cm,E} + \mathbf{C}_{EB} \underline{\omega}_B^\times \underline{s}_B) dm \\
 &= \mathbf{C}_{EB} \int_B (\underline{s}_B^\times \underline{v}_{cm,B} + \underline{s}_B^\times \underline{\omega}_B^\times \underline{s}_B) dm \\
 &= \mathbf{C}_{EB} \underbrace{\int_B \underline{s}_B^\times \underline{v}_{cm,B} dm}_{= \underline{0}} - \mathbf{C}_{EB} \int_B \underline{s}_B^\times \underline{s}_B^\times \underline{\omega}_B dm \\
 &= \mathbf{C}_{EB} \underbrace{\left[\int_B -\underline{s}_B^\times \underline{s}_B^\times dm \right]}_{\mathbf{I}_B} \underline{\omega}_B = \mathbf{C}_{EB} \mathbf{I}_B \underline{\omega}_B
 \end{aligned} \tag{5}$$

where $\mathbf{I}_B$ is called the *moment of inertia*.

Its matrix expression under a given (body-fixed) frame is:

$$\begin{aligned} \mathbf{I}_B &\triangleq \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \\ &= \begin{bmatrix} \int_V \rho(y^2 + z^2) dV & -\int_V \rho xy dV & -\int_V \rho xz dV \\ -\int_V \rho yx dV & \int_V \rho(z^2 + x^2) dV & -\int_V \rho yz dV \\ -\int_V \rho zx dV & -\int_V \rho zy dV & \int_V \rho(x^2 + y^2) dV \end{bmatrix} \end{aligned} \quad (6)$$

where ρ is the body density, $\underline{\mathbf{s}}_B = [x \ y \ z]^T$. $\mathbf{I}_B$ is defined in $\mathcal{F}_B$ as a constant matrix.

Angular Dynamics

Applying Newton's Second Law, the time-derivative of angular momentum equals the total net applied torque.

$$\begin{aligned}\underline{\tau}_{cm,E} &= \dot{\underline{h}}_{cm,E} \\ &= \frac{d}{dt} \mathbf{C}_{EB} \mathbf{I}_B \underline{\omega}_B = \mathbf{C}_{EB} \mathbf{I}_B \dot{\underline{\omega}}_B + \mathbf{C}_{EB} \underline{\omega}_B^\times \mathbf{I}_B \underline{\omega}_B\end{aligned}\quad (7)$$

Multiplying $\mathbf{C}_{BE}$ on both sides, we have:

$$\underline{\tau}_{cm,B} = \mathbf{I}_B \dot{\underline{\omega}}_B + \underline{\omega}_B^\times \mathbf{I}_B \underline{\omega}_B \quad (8)$$

Summarize: Rigid-body dynamics in $\mathcal{F}_B$

States:

$$\underline{\omega}_B = [p \ q \ r]^T \quad (9)$$

$$\underline{v}_B = [u \ v \ w]^T \quad (10)$$

Force and torque at cm:

$$\underline{f}_B = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \quad \underline{\tau}_B = \underline{\tau}_{cm,B} = \begin{bmatrix} L \\ M \\ N \end{bmatrix} \quad (11)$$

Plugging (3), we have the following translational equations of motion:

$$\begin{aligned} X &= m(\dot{u} + qw - rv) \\ Y &= m(\dot{v} + ru - pw) \\ Z &= m(\dot{w} + pv - qu) \end{aligned} \quad (12)$$

Plugging (8), we have the following angular equations of motion:

$$\begin{aligned} L &= I_{xx}\dot{p} - I_{xy}(\dot{q} - pr) - I_{xz}(\dot{r} + pq) - I_{yz}(q^2 - r^2) - (I_{yy} - I_{zz})qr \\ M &= I_{yy}\dot{q} - I_{yz}(\dot{r} - pq) - I_{yx}(\dot{p} + qr) - I_{zx}(r^2 - p^2) - (I_{zz} - I_{xx})pr \\ N &= I_{zz}\dot{r} - I_{zx}(\dot{p} - qr) - I_{zy}(\dot{q} + pr) - I_{xy}(p^2 - q^2) - (I_{xx} - I_{yy})pq \end{aligned} \quad (13)$$

2. Applied Flight Forces and Moments

- ▶ 6DOF motion variables to be solved by 6 sets of differential equations
- ▶ problem solved? Not yet because $\underline{f}_B$ and $\underline{\tau}_B$ are still unknown functions of the aircraft's motion. They come from three physical sources:
 - ▶ **aerodynamic forces/moments**: obtained from aerodynamic coefficients, e.g. $C_{L\alpha}$
 - ▶ **thrust force**: obtained from an engine model
 - ▶ **gravity**: the simplest of the three – just the mass times the gravitational acceleration, expressed in the inertial frame

Forces Must Be Resolved To $\mathcal{F}_B$

In vectorial form:

$$\underline{\mathbf{f}} = W\underline{\mathbf{z}}_E + T\underline{\mathbf{x}}_B - D\underline{\mathbf{x}}_W + Y\underline{\mathbf{y}}_W - L\underline{\mathbf{z}}_W, \quad (14)$$

where $\underline{\mathbf{z}}_E$ is the $\underline{\mathbf{z}}$ axis of the Earth-surface frame $\mathcal{F}_E$, $\underline{\mathbf{x}}_B$ is the $\underline{\mathbf{x}}$ axis of the body-fixed frame $\mathcal{F}_B$, $\underline{\mathbf{x}}_W, \underline{\mathbf{y}}_W, \underline{\mathbf{z}}_W$ are the axes of the air-trajectory frame $\mathcal{F}_W$, respectively. Resolving all forces to $\mathcal{F}_B$, we have:

$$\underline{\mathbf{f}}_B = \mathbf{C}_{BE} \begin{bmatrix} 0 \\ 0 \\ W \end{bmatrix} + \begin{bmatrix} T \\ 0 \\ 0 \end{bmatrix} + \mathbf{C}_{BW} \begin{bmatrix} -D \\ Y \\ -L \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad (15)$$

Question: why W not $-W$?

Total Moment in $\mathcal{F}_B$

Same convention as the total forces. Resolving all moments to $\mathcal{F}_B$, we have:

$$\underline{\tau}_B = \begin{bmatrix} L \\ M \\ N \end{bmatrix} \quad (16)$$

L here is the rolling moment, not lift¹ – always read it from which triad it appears in, $(X, Y, Z)/(D, Y, L)$ for forces vs. (L, M, N) for moments.

¹The symbol L is overloaded in flight dynamics: it denotes lift only when paired with drag D in a wind-axis force expression, and the rolling moment only when it appears in the moment triad (L, M, N) . See Caughey, *Introduction to Aircraft Stability and Control*, Cornell M&AE 5070 notes, p. 6.

Non-dimensional Aerodynamic Coefficients

$$L = \frac{1}{2}\rho V^2 S C_L, \quad C_L = C_{L_\alpha} \alpha + C_{L_{\delta_e}} \delta_e + C_{L_q} \hat{q} + \dots \quad (17)$$

Each term is a **stability or control derivative**: the sensitivity of a force/moment coefficient to one motion or control variable, e.g.

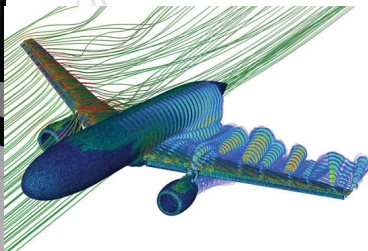
Coefficient	Definition	Physical meaning
C_{L_α}	$\partial C_L / \partial \alpha$	Lift-curve slope
C_{l_p}	$\partial C_l / \partial \hat{p}$	Roll damping
C_{D_α}	$\partial C_D / \partial \alpha$	Drag rise with angle of attack
$C_{L_{\delta_e}}$	$\partial C_L / \partial \delta_e$	Lift due to elevator deflection
C_{m_α}	$\partial C_m / \partial \alpha$	Pitch stiffness (longitudinal static stability)

Obtaining Aerodynamic Coefficients

Aerodynamic coefficients such as C_{L_α} are not derived analytically – they are obtained experimentally from wind tunnel testing, or computationally from CFD (Computational Fluid Dynamics) simulation.



(a) Wind tunnel test



(b) CFD simulation²

²Image sources: wind-tunnel photograph—NASA Ames Research Center, MD-11 12-ft Wind Tunnel Test (1996, public domain); CFD image—reproduced in A. Liu, *The Future of Flying: Explore More Energy-Efficient Airplane Wings*, Fig. 3(b) (2024).

From Derivatives to Coefficients: Longitudinal

Generalizing the C_L pattern from two slides ago, each coefficient is its zero-angle-of-attack value plus the sum of (derivative $\times$ variable):

$$C_L = C_{L_0} + C_{L_\alpha} \alpha + C_{L_q} \hat{q} + C_{L_{\dot{\alpha}}} \hat{\dot{\alpha}} + C_{L_{\delta_e}} \delta_e$$

$$C_D = C_{D_0} + C_{D_\alpha} \alpha + C_{D_{\delta_e}} \delta_e$$

$$C_m = C_{m_0} + C_{m_\alpha} \alpha + C_{m_q} \hat{q} + C_{m_{\dot{\alpha}}} \hat{\dot{\alpha}} + C_{m_{\delta_e}} \delta_e$$

C_D is usually taken independent of $\hat{q}$ and $\hat{\dot{\alpha}}$. The elevator contribution $C_{D_{\delta_e}} \delta_e$ may also be neglected when the available data show that it is sufficiently small.

From Derivatives to Coefficients: Lateral-Directional

The same pattern, built from the sideslip and roll/yaw rate derivatives instead:

$$C_Y = C_{Y_\beta}\beta + C_{Y_p}\hat{p} + C_{Y_r}\hat{r} + C_{Y_{\delta_a}}\delta_a + C_{Y_{\delta_r}}\delta_r$$

$$C_l = C_{l_\beta}\beta + C_{l_p}\hat{p} + C_{l_r}\hat{r} + C_{l_{\delta_a}}\delta_a + C_{l_{\delta_r}}\delta_r$$

$$C_n = C_{n_\beta}\beta + C_{n_p}\hat{p} + C_{n_r}\hat{r} + C_{n_{\delta_a}}\delta_a + C_{n_{\delta_r}}\delta_r$$

The cross-control derivatives $C_{l_{\delta_r}}$ and $C_{n_{\delta_a}}$ describe rudder-into-roll and aileron-into-yaw coupling. They are sometimes neglected when sufficiently small but are retained here for the general model. Once these coefficients are known numerically, they plug directly into the formulas on the next slide.

Computing Aerodynamic Forces and Moments from Coefficients

Once a coefficient is known numerically – from wind tunnel/CFD data, or built up from derivatives as on the earlier C_L example – the dimensional force or moment follows by multiplying back by the same normalization used to define it. Forces pick up one factor of the reference area S ; moments pick up an extra reference length ($\bar{c}$ for pitching, b for rolling/yawing):

$$\begin{aligned} D &= \frac{1}{2}\rho V^2 S C_D, & L &= \frac{1}{2}\rho V^2 S C_L, & Y &= \frac{1}{2}\rho V^2 S C_Y \\ L &= \frac{1}{2}\rho V^2 S b C_l, & M &= \frac{1}{2}\rho V^2 S \bar{c} C_m, & N &= \frac{1}{2}\rho V^2 S b C_n \end{aligned} \quad (18)$$

Aerodynamic Coefficients: Reference Frame and Point

Critical!

X, Y, Z, L, M, N – and therefore every derivative built from them, e.g. C_{m_α}, C_{l_p} – are defined

- ▶ in the **body-fixed frame** $\mathcal{F}_B$;
- ▶ about the **center of mass** of the aircraft *as configured during the wind-tunnel/CFD/flight test* that produced the data.

If the actual aircraft's cm differs from that reference point (e.g. a different loading), the moment coefficients must be transferred to the new cm before use – they do not automatically apply.

Normalized Angular and AOA Rates

The stability derivatives C_{m_q} , $C_{m_{\dot{\alpha}}}$, C_{l_p} , C_{n_r} , etc. are defined with respect to *normalized* (nondimensional) rates, not the raw rates in rad/s – this is what makes them dimensionless and comparable across aircraft of different size and speed. Longitudinal rates use the mean aerodynamic chord $\bar{c}$ as the reference length:

Rate	Dimensional	Normalized	Reference length
Pitch rate	q	$\hat{q} = \frac{q\bar{c}}{2U_e}$	$\bar{c}$ (mean chord)
AOA rate	$\dot{\alpha}$	$\hat{\alpha} = \frac{\dot{\alpha}\bar{c}}{2U_e}$	$\bar{c}$

Normalized Angular and AOA Rates (cont'd)

Lateral-directional rates use the wingspan b instead – both cases via the same time scale, $\bar{c}/2U_e$ or $b/2U_e$ respectively:

Rate	Dimensional	Normalized	Reference length
Roll rate	p	$\hat{p} = \frac{pb}{2U_e}$	b (span)
Yaw rate	r	$\hat{r} = \frac{rb}{2U_e}$	b

A Note on U_e : Linear vs. Nonlinear Simulation

- ▶ **Linearized (small-perturbation) model:** U_e is a fixed constant – the one trim airspeed the model was linearized about. The stability derivatives were measured/computed at that condition, and the whole framework assumes $V(t) \approx U_e$.
- ▶ **Full nonlinear 6DOF simulation:** replace U_e with the **instantaneous airspeed** $V(t)$ everywhere – both in the dynamic pressure and in the rate normalization:

$$\bar{q}(t) = \frac{1}{2}\rho V(t)^2, \quad \hat{q}(t) = \frac{q(t)\bar{c}}{2V(t)}$$

This reduces exactly to the linear model when $V(t) \approx U_e$, while still giving a sensible answer further from trim.

Caveats: the derivatives are only valid near the reference condition they were measured at (far excursions need derivative lookup tables vs. Mach/ α); and $\hat{q} = q\bar{c}/(2V)$ is numerically ill-behaved as $V \rightarrow 0$ – guard with a minimum-airspeed floor in code.

Thrust Model

- ▶ Depends on the engine type, e.g. turbojet, turbofan, propeller, etc.
- ▶ A function of the normalized propulsion command δ_p , the airspeed V (or Mach number M), and the air density ρ . These data are usually obtained from the engine manufacturer's performance charts.

Example: a simple turbojet engine thrust model:

$$T = T_{max} \delta_p \left(1 - \frac{V}{V_{max}} \right) \left(\frac{\rho}{\rho_0} \right) \quad (19)$$

where T_{max} is the maximum thrust at sea level, δ_p is the normalized propulsion command, V_{max} is the maximum airspeed for the engine, and ρ_0 is the air density at sea level.

Thrust Model Cont'd

- ▶ The thrust line is typically assumed to pass through the aircraft's center of mass, so the thrust force produces no moment.
- ▶ For engines mounted high or low relative to the center of mass (e.g. podded engines above/below the fuselage), the thrust line is offset from the center of mass, and this engine moment must be accounted for in the moment equations.

Summary: Forces and Moments

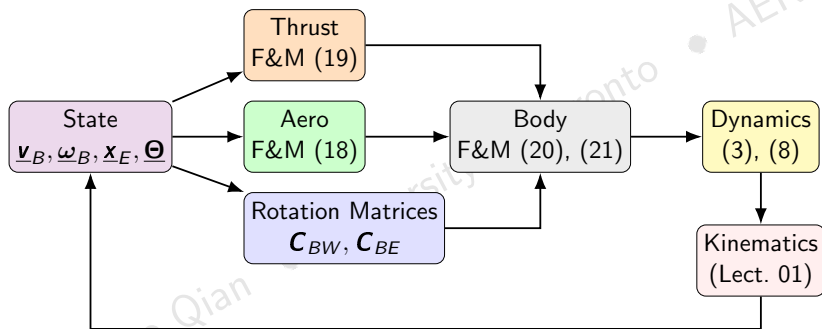
The total forces and moments acting on the aircraft in the body-fixed frame $\mathcal{F}_B$ are given by:

$$\underline{\mathbf{f}}_B = \mathbf{C}_{BE} \begin{bmatrix} 0 \\ 0 \\ W \end{bmatrix} + \begin{bmatrix} T \\ 0 \\ 0 \end{bmatrix} + \mathbf{C}_{BW} \begin{bmatrix} -D \\ Y \\ -L \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad (20)$$

$$\underline{\mathbf{\tau}}_B = \begin{bmatrix} L \\ M \\ N \end{bmatrix} \quad (21)$$

Combining these with the rigid-body dynamics equations in (3) and (8), we have a complete set of 6DOF nonlinear ODEs that can be solved numerically to simulate the aircraft's motion.

6DOF Nonlinear Simulation Diagram



C_{BE}, C_{BW} : rotation matrices from the Earth and wind frames, respectively, to the body frame

Flight Simulation is Fun!



(a) Microsoft Flight Simulator



(b) DCS World



(c) Simulator cockpit, inside view



(d) Simulator rig, outside view

3. Flight Dynamics: Linearization

- ▶ Since we already have nonlinear 6DOF equations of motion, why do we need linearization? Answer: linearization is a powerful tool for analysis and control design.
- ▶ How to linearize? The standard approach is to use the small perturbation principle, which assumes that the aircraft is flying in a steady flight condition, and the motion variables are small perturbations about this reference state.
- ▶ The longitudinal and lateral-directional motions are decoupled and can be analyzed separately.

Linearization and the Small Perturbation Principle

- ▶ Find an equilibrium flight condition, e.g. straight-and-level flight, a steady climb, or a steady turn.
- ▶ Longitudinal and lateral-directional motions are decoupled, and can be analyzed separately.
- ▶ Use small perturbation principle to linearize the nonlinear equations of motion about the reference flight condition.

Longitudinal Equations of Motion

Based on the rigid-body dynamics equations in (3) and (8), the longitudinal equations of motion are given by:

$$\begin{aligned} X - mg \sin \theta &= m(\dot{u} + qw - rv) \\ Z + mg \cos \theta \cos \phi &= m(\dot{w} + pv - qu) \\ M &= I_{yy}\dot{q} - I_{yz}(\dot{r} - pq) - I_{yx}(\dot{p} + qr) \\ &\quad - I_{zx}(r^2 - p^2) - (I_{zz} - I_{xx})pr \end{aligned} \tag{22}$$

Longitudinal Equilibrium Flight Condition

For steady, level flight, we have the following steady-state conditions:

$$\begin{aligned}u_e &= U_e \cos \theta_e, & v_e &= 0, & w_e &= U_e \sin \theta_e, \\p_e &= 0, & q_e &= 0, & r_e &= 0, \\ \theta &= \theta_e, & \psi_e &= 0, & \phi_e &= 0\end{aligned}\tag{23}$$

For steady level flight, the equilibrium angle of attack α_e is given by:

$$\alpha_e = \theta_e\tag{24}$$

why?

Longitudinal Equilibrium Flight Condition

In addition, the accelerations are zero:

$$\begin{aligned}\dot{u}_e &= 0, & \dot{v}_e &= 0, & \dot{w}_e &= 0, \\ \dot{p}_e &= 0, & \dot{q}_e &= 0, & \dot{r}_e &= 0\end{aligned}\tag{25}$$

In $\mathcal{F}_B$, treat $\phi_e = 0$, so the equilibrium forces and moments are given by:

$$\begin{aligned}X_e - mg \sin \theta_e &= 0 \\ Z_e + mg \cos \theta_e &= 0 \\ M_e &= 0\end{aligned}\tag{26}$$

with the aircraft holding these steady (trim) states.

Calculate Trim Conditions

In addition, for steady and level flight, $\mathbf{C}_{EW} = \mathbf{1}$.

$$W = L_e + T_e \sin \theta_e, \quad D_e = T_e \cos \theta_e, \quad M_e = 0 \quad (27)$$

Define the equilibrium dynamic pressure $Q_e = \frac{1}{2} \rho U_e^2$. Hence, we have the following trim conditions:

$$\begin{aligned} C_{L_0} + C_{L_\alpha} \alpha_e + C_{L_{\delta_e}} \delta_{e_e} + \frac{T_e}{Q_e S} \sin \theta_e - \frac{W}{Q_e S} &= 0 \\ C_{D_0} + C_{D_\alpha} \alpha_e + C_{D_{\delta_e}} \delta_{e_e} - \frac{T_e}{Q_e S} \cos \theta_e &= 0 \\ C_{m_0} + C_{m_\alpha} \alpha_e + C_{m_{\delta_e}} \delta_{e_e} &= 0 \end{aligned} \quad (28)$$

Because $\theta_e = \alpha_e$ enters nonlinearly through $\sin \theta_e, \cos \theta_e$, this system has no closed-form solution in general – solve it numerically for $(\alpha_e, \delta_{e_e}, T_e)$, e.g. with MATLAB's `fsolve` nonlinear equation solver.

Calculate Trim Conditions: Simplified Case

$(T_e \sin \theta_e \ll W)$

Assume $T_e \sin \theta_e \ll W$ (thrust's vertical component negligible compared to weight, e.g. cruise) – this decouples (α_e, δ_{e_e}) from T_e , with $C_{L_e} \approx W/(Q_e S)$:

$$C_{L_e} = C_{L_0} + C_{L_\alpha} \alpha_e + C_{L_{\delta_e}} \delta_{e_e} \quad (29)$$

$$0 = C_{m_0} + C_{m_\alpha} \alpha_e + C_{m_{\delta_e}} \delta_{e_e} \quad (30)$$

leading to

$$\begin{bmatrix} C_{L_\alpha} & C_{L_{\delta_e}} \\ C_{m_\alpha} & C_{m_{\delta_e}} \end{bmatrix} \begin{bmatrix} \alpha_e \\ \delta_{e_e} \end{bmatrix} = \begin{bmatrix} C_{L_e} - C_{L_0} \\ -C_{m_0} \end{bmatrix} \quad (31)$$

Calculate Trim Conditions: Simplified Case (cont'd)

The angle of attack and elevator angle to trim follow by Cramer's rule, with $\det = C_{L_\alpha} C_{m_{\delta_e}} - C_{L_{\delta_e}} C_{m_\alpha}$:

$$\alpha_e = \frac{C_{m_0} C_{L_{\delta_e}} + (C_{L_e} - C_{L_0}) C_{m_{\delta_e}}}{\det} \quad (32)$$

$$\delta_{e_e} = - \frac{C_{m_0} C_{L_\alpha} + C_{m_\alpha} (C_{L_e} - C_{L_0})}{\det} \quad (33)$$

Small Perturbation Principle

- ▶ The aircraft is flying in a steady flight condition. For a small perturbation about this reference state, the motion variables can be expressed as the sum of the equilibrium value and a small perturbation.
- ▶ For longitudinal motion, define the perturbation variables as follows:

Small Force and Moment Perturbations:

$$X = X_e + \Delta X$$

$$Z = Z_e + \Delta Z$$

$$M = \underbrace{M_e}_0 + \Delta M = \Delta M$$

Small Motion Perturbations:

$$\begin{aligned}
 u &= U_e \cos \theta_e + \Delta u, & v &= 0 + \Delta v, & w &= U_e \sin \theta_e + \Delta w, \\
 p &= \underbrace{p_e}_0 + \Delta p = \Delta p, & q &= \underbrace{q_e}_0 + \Delta q = \Delta q, & r &= \underbrace{r_e}_0 + \Delta r = \Delta r \\
 \theta &= \theta_e + \Delta \theta, & \phi &= \underbrace{\phi_e}_0 + \Delta \phi = \Delta \phi, & \psi &= \underbrace{\psi_e}_0 + \Delta \psi = \Delta \psi
 \end{aligned}$$

Since $U_e, \theta_e, p_e, q_e, r_e, \phi_e, \psi_e$ are all constants at the trim/equilibrium condition, their time derivatives vanish, leaving only the perturbation to differentiate:

$$\begin{aligned}
 \dot{u} &= \Delta \dot{u}, & \dot{v} &= \Delta \dot{v}, & \dot{w} &= \Delta \dot{w} \\
 \dot{p} &= \Delta \dot{p}, & \dot{q} &= \Delta \dot{q}, & \dot{r} &= \Delta \dot{r} \\
 \dot{\theta} &= \Delta \dot{\theta}, & \dot{\phi} &= \Delta \dot{\phi}, & \dot{\psi} &= \Delta \dot{\psi}
 \end{aligned} \tag{34}$$

Properties of Small Perturbations

$$\begin{aligned}\sin(\theta_e + \Delta\theta) &= \sin\theta_e \underbrace{\cos\Delta\theta}_{\approx 1} + \cos\theta_e \underbrace{\sin\Delta\theta}_{\approx \Delta\theta} \\ &\approx \sin\theta_e + \Delta\theta \cos\theta_e\end{aligned}\quad (35)$$

$$\begin{aligned}\cos(\theta_e + \Delta\theta) &= \cos\theta_e \underbrace{\cos\Delta\theta}_{\approx 1} - \sin\theta_e \underbrace{\sin\Delta\theta}_{\approx \Delta\theta} \\ &\approx \cos\theta_e - \Delta\theta \sin\theta_e\end{aligned}\quad (36)$$

For small perturbations, we can neglect the higher order terms, e.g. $\Delta\theta^2 \approx 0$. In general, treat any product of two or more small-perturbation quantities as zero, e.g. $\Delta u \Delta\alpha \approx 0$.

Applying the Small Perturbation Principle

According to (22), for longitudinal motion, treat $\phi_e = 0$. Hence, in the x direction, we have:

$$\begin{aligned}
 X_e + \Delta X - mg \underbrace{\sin(\theta_e + \Delta\theta)}_{\approx \sin \theta_e + \Delta\theta \cos \theta_e} \\
 = m \left(\Delta\dot{u} + \underbrace{U_e \sin \theta_e \Delta q}_{\text{first order}} + \underbrace{\Delta q \Delta w}_{\approx 0} - \underbrace{\Delta r \Delta v}_{\approx 0} \right) \quad (37)
 \end{aligned}$$

Using the equilibrium condition in (26), we have:

$$\underbrace{X_e - mg \sin \theta_e}_{=0} + \Delta X - mg \cos \theta_e \Delta\theta = m (\Delta\dot{u} + U_e \sin \theta_e \Delta q) \quad (38)$$

Hence, the linearized longitudinal equation in the x direction is given by:

$$\Delta X - mg \cos \theta_e \Delta\theta = m (\Delta\dot{u} + U_e \sin \theta_e \Delta q) \quad (39)$$

Applying the Small Perturbation Principle

Following the same procedure, in the z direction, we have:

$$\begin{aligned}
 & Z_e + \Delta Z + mg \underbrace{\cos(\theta_e + \Delta\theta)}_{\cos\theta_e - \Delta\theta \sin\theta_e} \\
 &= m(\Delta\dot{w} + \underbrace{\Delta p \Delta v}_{\approx 0} - \Delta q(U_e \cos\theta_e + \Delta u))
 \end{aligned} \tag{40}$$

Note that $\Delta q \Delta u \approx 0$, we have the following:

$$\underbrace{Z_e + mg \cos\theta_e}_{=0} + \Delta Z - mg \sin\theta_e \Delta\theta = m(\Delta\dot{w} - U_e \cos\theta_e \Delta q) \tag{41}$$

Applying the Small Perturbation Principle

Hence, the linearized longitudinal equation in the z direction is given by:

$$\Delta Z - mg \sin \theta_e \Delta \theta = m(\Delta \dot{w} - U_e \cos \theta_e \Delta q) \quad (42)$$

For the pitching moment, we have:

$$\underbrace{M_e}_{=0} + \Delta M = I_{yy} \Delta \dot{q} - \underbrace{I_{yz}}_{=0} (\dot{r} - pq) - \underbrace{I_{yx}}_{=0} (\dot{p} + qr) - I_{zx} (\underbrace{\Delta r^2}_{\approx 0} - \underbrace{\Delta p^2}_{\approx 0}) - (I_{zz} - I_{xx}) \underbrace{\Delta p \Delta r}_{\approx 0} \quad (43)$$

Hence, the linearized pitching moment equation is given by:

$$\Delta M = I_{yy} \Delta \dot{q} \quad (44)$$

Why $I_{yx} = 0$ for conventional aircraft?

- ▶ Conventional aircraft have **bilateral (left-right) mass symmetry** about the x-z plane: every mass element on the $+y$ side of the fuselage is mirrored by one on the $-y$ side.
- ▶ $I_{yx} = \int_V \rho xy dV$ and $I_{yz} = \int_V \rho yz dV$ each pair the symmetry-plane coordinate (x or z) with y – for every mass element at $+y$ contributing $+xy$ (or $+yz$), its mirror at $-y$ contributes the exact negative, so the integral cancels to zero *identically*.
- ▶ $I_{xz} = \int_V \rho xz dV$ survives, since x and z both lie *in* the symmetry plane – nothing cancels it.
- ▶ This is a geometric property of the mass distribution, true even in the full nonlinear equations – it is what allows a wings-level, zero-sideslip trim to exist and longitudinal/lateral motion to decouple to first order.

Applying the Small Perturbation Principle

From lecture 1, the Euler rate equation is:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \mathbf{S}_B^{-1} \underline{\omega}_B = \begin{bmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi \sec \theta & \cos \phi \sec \theta \end{bmatrix} \underline{\omega}_B$$

$$\dot{\theta} = \cos \phi q - \sin \phi r$$

Applying the equilibrium condition $\phi_e = 0$, we have:

$$\Delta \dot{\theta} = \Delta q$$

Expressing Small Aerodynamic Force and Moment Perturbations Using Stability Derivatives

$$\begin{bmatrix} \Delta X \\ \Delta Z \\ \Delta M \end{bmatrix} = \begin{bmatrix} X_u & X_w & X_q & X_{\dot{u}} & X_{\dot{w}} & X_{\dot{q}} \\ Z_u & Z_w & Z_q & Z_{\dot{u}} & Z_{\dot{w}} & Z_{\dot{q}} \\ M_u & M_w & M_q & M_{\dot{u}} & M_{\dot{w}} & M_{\dot{q}} \end{bmatrix} \cdot \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \dot{u} \\ \Delta \dot{w} \\ \Delta \dot{q} \end{bmatrix} + \begin{bmatrix} X_{\delta_e} & X_{\delta_p} \\ Z_{\delta_e} & Z_{\delta_p} \\ M_{\delta_e} & M_{\delta_p} \end{bmatrix} \begin{bmatrix} \delta_e \\ \delta_p \end{bmatrix}$$

Reorganizing the equations into matrix form and dropping the Δ notation, with all variables understood to be perturbations from their equilibrium values, gives the following linearized longitudinal equations of motion:

$$\underbrace{\begin{bmatrix} -X_u & -X_w & mU_e \sin \theta_e - X_q & mg \cos \theta_e \\ -Z_u & -Z_w & -(mU_e \cos \theta_e + Z_q) & mg \sin \theta_e \\ -M_u & -M_w & -M_q & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}}_{A'} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \underbrace{\begin{bmatrix} (m - X_{\dot{u}}) & -X_{\dot{w}} & -X_{\dot{q}} & 0 \\ -Z_{\dot{u}} & (m - Z_{\dot{w}}) & -Z_{\dot{q}} & 0 \\ -M_{\dot{u}} & -M_{\dot{w}} & (I_{yy} - M_{\dot{q}}) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_E \begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \underbrace{\begin{bmatrix} X_{\delta_e} & X_{\delta_p} \\ Z_{\delta_e} & Z_{\delta_p} \\ M_{\delta_e} & M_{\delta_p} \\ 0 & 0 \end{bmatrix}}_{B'} \begin{bmatrix} \delta_e \\ \delta_p \end{bmatrix} \quad (45)$$

Rate derivatives are treated as zero (or negligible compared with other terms) except $Z_{\dot{w}}$ and $M_{\dot{w}}$, as they are related to the unsteady change rate of the angle of attack α .

- ▶ Zero: $X_{\dot{u}}$, $X_{\dot{w}}$, $X_{\dot{q}}$, $Z_{\dot{u}}$, $Z_{\dot{q}}$, $M_{\dot{u}}$, $M_{\dot{q}}$
- ▶ Nonzero: $Z_{\dot{w}}$, $M_{\dot{w}}$

$$\mathbf{E} = \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m - Z_{\dot{w}} & 0 & 0 \\ 0 & -M_{\dot{w}} & I_{yy} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\mathbf{E}$ becomes a block diagonal matrix, with the (w, q) block being a 2×2 matrix that can be inverted analytically to yield the final state-space form of the linearized longitudinal equations of motion.

Closed-Form Inverse of a 2×2 Matrix

For a general 2×2 matrix,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad (46)$$

With rate derivatives dropped except $Z_{\dot{w}}$, $M_{\dot{w}}$, the (w, q) block of the mass matrix on the previous slide is exactly of this form, with $a = m - Z_{\dot{w}}$, $b = 0$, $c = -M_{\dot{w}}$, $d = I_{yy}$:

$$\begin{bmatrix} m - Z_{\dot{w}} & 0 \\ -M_{\dot{w}} & I_{yy} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{m - Z_{\dot{w}}} & 0 \\ \frac{M_{\dot{w}}}{I_{yy}(m - Z_{\dot{w}})} & \frac{1}{I_{yy}} \end{bmatrix} \quad (47)$$

since $\det = (m - Z_{\dot{w}})I_{yy} - 0 \cdot (-M_{\dot{w}}) = I_{yy}(m - Z_{\dot{w}})$. This is exactly the factor that turns up dividing every $Z_{\bullet}$ and $M_{\bullet}$ term in $\mathbf{A}_{long}$ on the next slide.

$$\mathbf{E}^{-1} = \begin{bmatrix} 1/m & 0 & 0 & 0 \\ 0 & \frac{1}{m - Z_{\dot{w}}} & 0 & 0 \\ 0 & \frac{M_{\dot{w}}}{I_{yy}(m - Z_{\dot{w}})} & \frac{1}{I_{yy}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (48)$$

The linearized longitudinal equations can then be written in state-space form as

$$\dot{\underline{x}} = \mathbf{A}_{long} \underline{x} + \mathbf{B}_{long} \underline{u} \quad (49)$$

where $\underline{x} = [u \ w \ q \ \theta]^T$, $\underline{u} = [\delta_e \ \delta_p]^T$. System matrices $\mathbf{A}_{long}$ and $\mathbf{B}_{long}$ are given by:

$$\mathbf{A}_{long} = -\mathbf{E}^{-1} \mathbf{A}', \quad \mathbf{B}_{long} = \mathbf{E}^{-1} \mathbf{B}'$$

Longitudinal Linearized Model

$$\mathbf{A}_{long} =$$

$$\begin{bmatrix} \frac{\frac{X_u}{m}}{m-Z_{\dot{w}}} & \frac{\frac{X_w}{m}}{m-Z_{\dot{w}}} & \frac{\frac{X_q}{m} - U_e \sin \theta_e}{Z_q + m U_e \cos \theta_e} & -g \cos \theta_e \\ \frac{1}{I_{yy}} \left(M_u + \frac{Z_u}{m-Z_{\dot{w}}} M_{\dot{w}} \right) & \frac{1}{I_{yy}} \left(M_w + \frac{Z_w}{m-Z_{\dot{w}}} M_{\dot{w}} \right) & \frac{1}{I_{yy}} \left(M_q + \frac{Z_q + m U_e \cos \theta_e}{m-Z_{\dot{w}}} M_{\dot{w}} \right) & -\frac{1}{I_{yy}} \left(\frac{mg \sin \theta_e}{m-Z_{\dot{w}}} M_{\dot{w}} \right) \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (50)$$

$$\mathbf{B}_{long} =$$

$$\begin{bmatrix} \frac{\frac{X_{\delta_e}}{m}}{m-Z_{\dot{w}}} & \frac{\frac{X_{\delta_p}}{m}}{m-Z_{\dot{w}}} \\ \frac{1}{I_{yy}} \left(M_{\delta_e} + \frac{Z_{\delta_e}}{m-Z_{\dot{w}}} M_{\dot{w}} \right) & \frac{1}{I_{yy}} \left(M_{\delta_p} + \frac{Z_{\delta_p}}{m-Z_{\dot{w}}} M_{\dot{w}} \right) \\ 0 & 0 \end{bmatrix}$$

Lateral Linearized Model

Recall the lateral equilibrium flight conditions:

$$\begin{aligned}
 u &= U_e \cos \theta_e + \Delta u, & v &= 0 + \Delta v, & w &= U_e \sin \theta_e + \Delta w, \\
 p &= \underbrace{p_e}_0 + \Delta p = \Delta p, & q &= \underbrace{q_e}_0 + \Delta q = \Delta q, & r &= \underbrace{r_e}_0 + \Delta r = \Delta r \\
 \theta &= \theta_e + \Delta \theta, & \phi &= \underbrace{\phi_e}_0 + \Delta \phi = \Delta \phi, & \psi &= \underbrace{\psi_e}_0 + \Delta \psi = \Delta \psi
 \end{aligned}$$

Since $U_e, \theta_e, p_e, q_e, r_e, \phi_e, \psi_e$ are all constants at the trim/equilibrium condition, their time derivatives vanish, leaving only the perturbation to differentiate:

$$\begin{aligned}
 \dot{u} &= \Delta \dot{u}, & \dot{v} &= \Delta \dot{v}, & \dot{w} &= \Delta \dot{w} \\
 \dot{p} &= \Delta \dot{p}, & \dot{q} &= \Delta \dot{q}, & \dot{r} &= \Delta \dot{r} \\
 \dot{\theta} &= \Delta \dot{\theta}, & \dot{\phi} &= \Delta \dot{\phi}, & \dot{\psi} &= \Delta \dot{\psi}
 \end{aligned} \tag{51}$$

Lateral Equilibrium Flight Condition

The lateral-directional linearized equations of motion can be derived using the same procedure as for longitudinal motion. Applying the small perturbation principle gives

$$\begin{aligned}Y + mg \cos \theta \sin \phi &= m(\dot{v} + ru - pw) \\L &= I_{xx}\dot{p} - I_{xz}(\dot{r} + pq) - (I_{yy} - I_{zz})qr \\N &= I_{zz}\dot{r} - I_{zx}(\dot{p} - qr) - (I_{xx} - I_{yy})pq\end{aligned}$$

The equilibrium flight condition is given by:

$$\begin{aligned}Y_e &= 0 \\L_e &= 0 \\N_e &= 0\end{aligned}\tag{52}$$

Applying the equilibrium condition in (52), we have:

$$\underbrace{Y_e}_{=0} + \Delta Y + mg \cos \theta_e \underbrace{\sin \Delta \phi}_{\approx \Delta \phi} = m(\Delta \dot{v} + \Delta r U_e \cos \theta_e - \Delta p U_e \sin \theta_e) \quad (53)$$

$$\Delta Y + mg \cos \theta_e \Delta \phi - m \Delta r U_e \cos \theta_e + m \Delta p U_e \sin \theta_e = m \Delta \dot{v} \quad (54)$$

$$L_e + \Delta L = I_{xx} \Delta \dot{p} - I_{xz} (\Delta \dot{r} + \underbrace{\Delta p \Delta q}_{\approx 0}) - (I_{yy} - I_{zz}) \underbrace{\Delta q \Delta r}_{\approx 0} \quad (55)$$

$$\Delta L = I_{xx} \Delta \dot{p} - I_{xz} \Delta \dot{r}$$

For yaw motion, we have

$$N_e + \Delta N = I_{zz}\Delta\dot{r} - I_{xz}(\Delta\dot{p} - \underbrace{\Delta q\Delta r}_{\approx 0}) - (I_{xx} - I_{yy})\underbrace{\Delta p\Delta q}_{\approx 0} \quad (56)$$

$$\Delta N = I_{zz}\Delta\dot{r} - I_{xz}\Delta\dot{p}$$

The roll motion equation is:

$$\dot{\phi} = p + \sin\phi \tan\theta_e q + \cos\phi \tan\theta_e r \quad (57)$$

The small perturbation of roll motion is:

$$\begin{aligned} \Delta\dot{\phi} &= \Delta p + \underbrace{\sin\Delta\phi}_{\approx\Delta\phi} \tan\theta_e \Delta q + \underbrace{\cos\Delta\phi}_{\approx 1} \tan\theta_e \Delta r \\ &\approx \Delta p + \tan\theta_e \Delta r \end{aligned} \quad (58)$$

The yaw motion equation is:

$$\dot{\psi} = \sin \phi \sec \theta q + \cos \phi \sec \theta r \quad (59)$$

The small perturbation of yaw motion is:

$$\Delta \dot{\psi} = \underbrace{\sin(\Delta \phi)}_{\approx \Delta \phi} \sec \theta_e \Delta q + \underbrace{\cos \Delta \phi}_{\approx 1} \sec \theta_e \Delta r \approx \sec \theta_e \Delta r \quad (60)$$

Similarly, the lateral aerodynamic derivative terms are applied to lateral force, moments and control,

$$\begin{bmatrix} \Delta Y \\ \Delta L \\ \Delta N \end{bmatrix} = \begin{bmatrix} Y_v & Y_p & Y_r & Y_{\dot{v}} & Y_{\dot{p}} & Y_{\dot{r}} \\ L_v & L_p & L_r & L_{\dot{v}} & L_{\dot{p}} & L_{\dot{r}} \\ N_v & N_p & N_r & N_{\dot{v}} & N_{\dot{p}} & N_{\dot{r}} \end{bmatrix} \cdot \begin{bmatrix} v \\ p \\ r \\ \dot{v} \\ \dot{p} \\ \dot{r} \end{bmatrix} \quad (61)$$

$$+ \begin{bmatrix} Y_{\delta_a} & Y_{\delta_r} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix} \quad (62)$$

Lateral Small Perturbation in Matrix Form

$$\underbrace{\begin{bmatrix} -Y_v & -Y_p - mU_e \sin \theta_e & mU_e \cos \theta_e - Y_r & -mg \cos \theta_e \\ -L_v & -L_p & -L_r & 0 \\ -N_v & -N_p & -N_r & 0 \\ 0 & -1 & -\tan \theta_e & 0 \end{bmatrix}}_{A'} \begin{bmatrix} v \\ p \\ r \\ \phi \end{bmatrix} + \underbrace{\begin{bmatrix} (m - Y_{\dot{v}}) & -Y_{\dot{p}} & -Y_{\dot{r}} & 0 \\ -L_{\dot{v}} & (I_{xx} - L_{\dot{p}}) & -(I_{xz} + L_{\dot{r}}) & 0 \\ -N_{\dot{v}} & -(I_{xz} + N_{\dot{p}}) & (I_{zz} - N_{\dot{r}}) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_E \begin{bmatrix} \dot{v} \\ \dot{p} \\ \dot{r} \\ \dot{\phi} \end{bmatrix} = \underbrace{\begin{bmatrix} Y_{\delta_a} & Y_{\delta_r} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \\ 0 & 0 \end{bmatrix}}_{B'} \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix}$$

Rate derivatives are set to zero (or treated as negligible compared with other terms) in the lateral motion – unlike the longitudinal case, there is no comparably significant unsteady effect (such as downwash lag) that motivates keeping any of them.

► Zero: $Y_{\dot{v}}, Y_{\dot{p}}, Y_{\dot{r}}, L_{\dot{v}}, L_{\dot{p}}, L_{\dot{r}}, N_{\dot{v}}, N_{\dot{p}}, N_{\dot{r}}$

$$\mathbf{E} = \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & I_{xx} & -I_{xz} & 0 \\ 0 & -I_{xz} & I_{zz} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\mathbf{E}$ is already block diagonal, with the (p, r) block being a 2×2 matrix that can be inverted analytically (the same closed-form 2×2 inverse used for the longitudinal case) to yield the final state-space form of the linearized lateral equations of motion.

Closed-form Inverse of the (p, r) Block

The (p, r) block of $\mathbf{E}$ is

$$\begin{bmatrix} I_{xx} & -I_{xz} \\ -I_{xz} & I_{zz} \end{bmatrix}^{-1} = \frac{1}{I_{xx}I_{zz} - I_{xz}^2} \begin{bmatrix} I_{zz} & I_{xz} \\ I_{xz} & I_{xx} \end{bmatrix} = \begin{bmatrix} I'_{xx} & I'_{xz} \\ I'_{xz} & I'_{zz} \end{bmatrix} \quad (63)$$

using $a = I_{xx}$, $b = -I_{xz}$, $c = -I_{xz}$, $d = I_{zz}$, so

$\det = I_{xx}I_{zz} - (-I_{xz})(-I_{xz}) = I_{xx}I_{zz} - I_{xz}^2$. This is exactly the $I'_{xx}, I'_{zz}, I'_{xz}$ combination that appears in every $L_{\bullet}$ and $N_{\bullet}$ term of $\mathbf{A}_{litr}$ on the next slide.

$$\mathbf{E}^{-1} = \begin{bmatrix} 1/m & 0 & 0 & 0 \\ 0 & I'_{xx} & I'_{xz} & 0 \\ 0 & I'_{xz} & I'_{zz} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (64)$$

where

$$I'_{xx} = \frac{I_{zz}}{I_{xx}I_{zz} - I_{xz}^2}, \quad I'_{zz} = \frac{I_{xx}}{I_{xx}I_{zz} - I_{xz}^2}, \quad I'_{xz} = \frac{I_{xz}}{I_{xx}I_{zz} - I_{xz}^2} \quad (65)$$

Setting the lateral rate derivatives to zero gives

$$\begin{bmatrix} m & 0 & 0 & 0 \\ 0 & I_{xx} & -I_{xz} & 0 \\ 0 & -I_{xz} & I_{zz} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{p} \\ \dot{r} \\ \dot{\phi} \end{bmatrix} = \underbrace{\begin{bmatrix} Y_v & Y_p + mU_e \sin \theta_e & Y_r - mU_e \cos \theta_e & mg \cos \theta_e \\ L_v & L_p & L_r & 0 \\ N_v & N_p & N_r & 0 \\ 0 & 1 & \tan \theta_e & 0 \end{bmatrix}}_{-A'} \begin{bmatrix} v \\ p \\ r \\ \phi \end{bmatrix} + \begin{bmatrix} Y_{\delta_a} & Y_{\delta_r} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix}$$

$$\mathbf{A}_{latr} = -\mathbf{E}^{-1}\mathbf{A}', \quad \mathbf{B}_{latr} = \mathbf{E}^{-1}\mathbf{B}'$$

Finally, the linearized lateral-directional equations can be represented by

$$\dot{\underline{x}} = \mathbf{A}_{latr} \underline{x} + \mathbf{B}_{latr} \underline{u} \quad (66)$$

where $\underline{x} = [v \ p \ r \ \phi]^T$, $\underline{u} = [\delta_a \ \delta_r]^T$,

$\mathbf{A}_{latr} =$

$$\begin{bmatrix} \frac{Y_v}{m} & \frac{Y_p}{m} + U_e \sin \theta_e & \frac{Y_r}{m} - U_e \cos \theta_e & g \cos \theta_e \\ I'_{xx} L_v + I'_{xz} N_v & I'_{xx} L_p + I'_{xz} N_p & I'_{xx} L_r + I'_{xz} N_r & 0 \\ I'_{zz} N_v + I'_{xz} L_v & I'_{zz} N_p + I'_{xz} L_p & I'_{zz} N_r + I'_{xz} L_r & 0 \\ 0 & 1 & \tan \theta_e & 0 \end{bmatrix} \quad (67)$$

$\mathbf{B}_{latr} =$

$$\begin{bmatrix} \frac{Y_{\delta_a}}{m} & \frac{Y_{\delta_r}}{m} \\ I'_{xx} L_{\delta_a} + I'_{xz} N_{\delta_a} & I'_{xx} L_{\delta_r} + I'_{xz} N_{\delta_r} \\ I'_{zz} N_{\delta_a} + I'_{xz} L_{\delta_a} & I'_{zz} N_{\delta_r} + I'_{xz} L_{\delta_r} \\ 0 & 0 \end{bmatrix} \quad (68)$$

Sometimes we use the following symbols for a simpler notation,

$$\mathbf{A}_{latr} = \begin{bmatrix} \mathcal{Y}_v & \mathcal{Y}_p & \mathcal{Y}_r & g \cos \theta_e \\ \mathcal{L}_v & \mathcal{L}_p & \mathcal{L}_r & 0 \\ \mathcal{N}_v & \mathcal{N}_p & \mathcal{N}_r & 0 \\ 0 & 1 & \tan \theta_e & 0 \end{bmatrix}, \quad \mathbf{B}_{latr} = \begin{bmatrix} \mathcal{Y}_{\delta_a} & \mathcal{Y}_{\delta_r} \\ \mathcal{L}_{\delta_a} & \mathcal{L}_{\delta_r} \\ \mathcal{N}_{\delta_a} & \mathcal{N}_{\delta_r} \\ 0 & 0 \end{bmatrix} \quad (69)$$

A_{latr}, B_{latr} : What the Script Symbols Mean

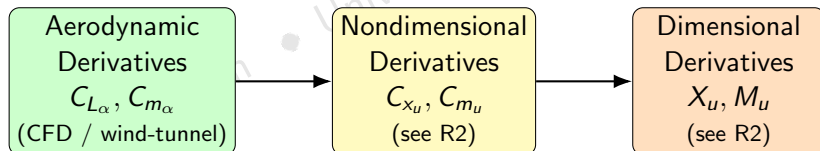
Each script symbol is shorthand for a specific combination of dimensional derivatives and the reduced-inertia terms $I'_{xx}, I'_{zz}, I'_{xz}$ (recall $I'_{xx} = I_{zz}/(I_{xx}I_{zz} - I_{xz}^2)$, etc.):

Symbol	Full expression	Symbol	Full expression
$\mathcal{Y}_v$	Y_v/m	$\mathcal{Y}_{\delta_a}$	Y_{δ_a}/m
$\mathcal{Y}_p$	$Y_p/m + U_e \sin \theta_e$	$\mathcal{Y}_{\delta_r}$	Y_{δ_r}/m
$\mathcal{Y}_r$	$Y_r/m - U_e \cos \theta_e$		
$\mathcal{L}_v$	$I'_{xx}L_v + I'_{xz}N_v$	$\mathcal{L}_{\delta_a}$	$I'_{xx}L_{\delta_a} + I'_{xz}N_{\delta_a}$
$\mathcal{L}_p$	$I'_{xx}L_p + I'_{xz}N_p$	$\mathcal{L}_{\delta_r}$	$I'_{xx}L_{\delta_r} + I'_{xz}N_{\delta_r}$
$\mathcal{L}_r$	$I'_{xx}L_r + I'_{xz}N_r$		
$\mathcal{N}_v$	$I'_{zz}N_v + I'_{xz}L_v$	$\mathcal{N}_{\delta_a}$	$I'_{zz}N_{\delta_a} + I'_{xz}L_{\delta_a}$
$\mathcal{N}_p$	$I'_{zz}N_p + I'_{xz}L_p$	$\mathcal{N}_{\delta_r}$	$I'_{zz}N_{\delta_r} + I'_{xz}L_{\delta_r}$
$\mathcal{N}_r$	$I'_{zz}N_r + I'_{xz}L_r$		

Note $\mathcal{Y}_p, \mathcal{Y}_r$ carry the extra $U_e \sin \theta_e, U_e \cos \theta_e$ terms (sideslip-velocity linearization); every $\mathcal{L}_\bullet, \mathcal{N}_\bullet$ mixes $L_\bullet$ and $N_\bullet$ via I'_{xz} – decouples only when $I_{xz} = 0$.

How to Obtain Dimensional Derivatives

- ▶ Since we have the linearized equations of motion, the final piece of the puzzle is to calculate dimensional derivatives such as Z_w using aerodynamic coefficients such as C_{L_α} .
- ▶ Intermediate step: stability derivatives such as C_{x_u}



Stability Derivatives

	C_x	C_z	C_m	C_y	C_l	C_n
e	C_{x_e}	C_{z_e}	C_{m_e}			
$\hat{u}$	C_{x_u}	C_{z_u}	C_{m_u}			
α	C_{x_α}	C_{z_α}	C_{m_α}			
$\hat{q}$	C_{x_q}	C_{z_q}	C_{m_q}			
$\hat{\dot{\alpha}}$	$C_{x_{\dot{\alpha}}}$	$C_{z_{\dot{\alpha}}}$	$C_{m_{\dot{\alpha}}}$			
δ_e	$C_{x_{\delta_e}}$	$C_{z_{\delta_e}}$	$C_{m_{\delta_e}}$			
β				C_{y_β}	C_{l_β}	C_{n_β}
$\hat{p}$				C_{y_p}	C_{l_p}	C_{n_p}
$\hat{r}$				C_{y_r}	C_{l_r}	C_{n_r}
$\hat{\dot{\beta}}$				$C_{y_{\dot{\beta}}}$	$C_{l_{\dot{\beta}}}$	$C_{n_{\dot{\beta}}}$
δ_r				$C_{y_{\delta_r}}$	$C_{l_{\delta_r}}$	$C_{n_{\delta_r}}$
δ_a				$C_{y_{\delta_a}}$	$C_{l_{\delta_a}}$	$C_{n_{\delta_a}}$

Hats on coefficient subscripts are omitted by convention; for example, $C_{x_{\dot{\alpha}}}$ denotes differentiation with respect to $\hat{\dot{\alpha}}$.

Non-dimensional State Variables

Longitudinal perturbations are normalized by the trim speed U_e and the mean aerodynamic chord $\bar{c}$:

Motion	Dimensional	Non-dimensional	Scale
Axial velocity	u	$\hat{u} = \frac{u}{U_e}$	U_e
Angle of attack	w	$\alpha \simeq \frac{w}{U_e}$	U_e
Pitch rate	q	$\hat{q} = \frac{q\bar{c}}{2U_e}$	$\frac{2U_e}{\bar{c}}$
AOA rate	$\dot{\alpha}$	$\hat{\dot{\alpha}} = \frac{\dot{\alpha}\bar{c}}{2U_e}$	$\frac{2U_e}{\bar{c}}$

$\alpha \simeq w/U_e$ uses the small-disturbance approximation about the trim condition.

Non-dimensional State Variables (cont'd)

Lateral-directional perturbations are normalized by the trim speed U_e and the wingspan b :

Motion	Dimensional	Non-dimensional	Scale
Sideslip	v	$\beta \simeq \frac{v}{U_e}$	U_e
Roll rate	p	$\hat{p} = \frac{pb}{2U_e}$	$\frac{2U_e}{b}$
Yaw rate	r	$\hat{r} = \frac{rb}{2U_e}$	$\frac{2U_e}{b}$
Sideslip rate	$\dot{\beta}$	$\hat{\dot{\beta}} = \frac{\dot{\beta}b}{2U_e}$	$\frac{2U_e}{b}$

$\beta \simeq v/U_e$ uses the small-disturbance approximation about the trim condition.

Force Coefficients

Following the NASA aerodynamic convention, the aerodynamic force and moment coefficients are defined as

$$C_X = \frac{X}{QS}, \quad C_Y = \frac{Y}{QS}, \quad C_Z = \frac{Z}{QS}$$
$$C_l = \frac{L}{QSb}, \quad C_m = \frac{M}{QS\bar{c}}, \quad C_n = \frac{N}{QSb}$$

where $Q = \frac{1}{2}\rho V^2$ is the dynamic pressure, S the reference (wing) area, $\bar{c}$ the mean aerodynamic chord, and b the wingspan.

Convention: X, Y, Z, L, M, N contain aerodynamic loads only. Thrust is represented separately by T or $C_T = T/(QS)$.

Thrust Model

Under a simplified quasi-steady assumption, the thrust magnitude is modeled as a function of longitudinal airspeed and throttle command:

$$T = T(U, \delta_p), \quad 0 \leq \delta_p \leq 1. \quad (70)$$

- ▶ T is the thrust magnitude. The thrust force is assumed to act along the positive body x -axis:

$$\underline{\mathbf{F}}_{T,B} = [T \quad 0 \quad 0]^T.$$

- ▶ U is the component of the air-relative velocity along the body x -axis. For small angles of attack and sideslip, $U \simeq V$.
- ▶ δ_p is the normalized propulsion command, from minimum/idle ($\delta_p = 0$) to full thrust ($\delta_p = 1$).

Notation. The assignments, the drone data sheet, and the supplied MATLAB files write this same throttle command as δ_T .

Thrust Derivatives

Let the dimensional thrust model be $T = T(U, \delta_p)$ and define

$$Q = \frac{1}{2}\rho U^2, \quad C_T = \frac{T}{QS}, \quad \hat{u} = \frac{u}{U_e}, \quad U = U_e + u.$$

At equilibrium,

$$Q_e = \frac{1}{2}\rho U_e^2, \quad C_{T_e} = \frac{T_e}{Q_e S}, \quad T_e = T(U_e, \delta_{p_e}).$$

Holding throttle fixed, define

$$T_u = \left. \frac{\partial T}{\partial U} \right|_e, \quad C_{T_u} = \left. \frac{\partial C_T}{\partial \hat{u}} \right|_e.$$

Since $\partial Q / \partial u|_e = 2Q_e / U_e$, the chain rule gives

$$C_{T_u} = \frac{U_e T_u}{Q_e S} - 2C_{T_e} \quad \Longleftrightarrow \quad T_u = \frac{Q_e S}{U_e} (C_{T_u} + 2C_{T_e}).$$

Equilibrium Stability Coefficients

Here C_{x_e} , C_{z_e} , C_{m_e} are aerodynamic coefficients only; thrust is kept separate following the NASA convention. Let

$$Q_e = \frac{1}{2} \rho U_e^2, \quad C_{W_e} = \frac{mg}{Q_e S}.$$

For steady longitudinal trim, the body-axis force and moment balances are

$$\begin{aligned} C_{x_e} + C_{T_e} - C_{W_e} \sin \theta_e &= 0, \\ C_{z_e} + C_{W_e} \cos \theta_e &= 0, \\ C_{m_e} &= 0, \end{aligned} \quad \Rightarrow \quad \begin{aligned} C_{x_e} &= C_{W_e} \sin \theta_e - C_{T_e}, \\ C_{z_e} &= -C_{W_e} \cos \theta_e. \end{aligned}$$

The same aerodynamic coefficients follow from the wind-to-body-axis rotation:

$$\begin{aligned} C_{x_e} &= C_{L_e} \sin \alpha_e - C_{D_e} \cos \alpha_e \simeq -C_{D_e}, \\ C_{z_e} &= -C_{L_e} \cos \alpha_e - C_{D_e} \sin \alpha_e \simeq -C_{L_e}. \end{aligned}$$

This assumes thrust acts along $+\underline{x}_B$ through the center of mass.

Stability Derivative Definitions

Stability derivatives are local slopes evaluated at the equilibrium condition, with all other independent variables held fixed. For example,

$$C_{x_u} \equiv \left. \frac{\partial C_x}{\partial (u/U_e)} \right|_e = U_e \left. \frac{\partial C_x}{\partial u} \right|_e,$$

$$C_{z_w} \equiv \left. \frac{\partial C_z}{\partial (w/U_e)} \right|_e = U_e \left. \frac{\partial C_z}{\partial w} \right|_e,$$

$$C_{z_\alpha} \equiv \left. \frac{\partial C_z}{\partial \alpha} \right|_e,$$

$$C_{z_q} \equiv \left. \frac{\partial C_z}{\partial (q\bar{c}/(2U_e))} \right|_e = \frac{2U_e}{\bar{c}} \left. \frac{\partial C_z}{\partial q} \right|_e.$$

Under the small-disturbance approximation $\alpha \simeq w/U_e$, $C_{z_w} \simeq C_{z_\alpha}$.

Here C_{z_w} is nondimensional; the dimensional force derivative is $Z_w = \partial Z / \partial w|_e$. These coefficient derivatives are aerodynamic only; C_{T_u} is kept separate under the NASA convention.

Longitudinal: General Formulas (1/2)

Differentiate the exact body-axis rotation

$$C_X = C_L \sin \alpha - C_D \cos \alpha, \quad C_Z = -C_L \cos \alpha - C_D \sin \alpha$$

at the actual trim angle α_e . For angle of attack,

$$C_{X_\alpha} = C_{L_\alpha} \sin \alpha_e + C_{L_e} \cos \alpha_e - C_{D_\alpha} \cos \alpha_e + C_{D_e} \sin \alpha_e$$

$$C_{Z_\alpha} = -C_{D_\alpha} \sin \alpha_e - C_{D_e} \cos \alpha_e - C_{L_\alpha} \cos \alpha_e + C_{L_e} \sin \alpha_e$$

$$C_{m_\alpha} = \left. \frac{\partial C_m}{\partial \alpha} \right|_e.$$

These expressions retain every $\sin \alpha_e$ and $\cos \alpha_e$ term and therefore apply at arbitrary trim angle of attack within the linearization.

Longitudinal: General Formulas (2/2)

For $\xi \in \{u, q, \dot{\alpha}, \delta_e\}$, where rate derivatives are taken with respect to their normalized variables, the exact rotation gives

$$\begin{aligned}C_{x_\xi} &= C_{L_\xi} \sin \alpha_e - C_{D_\xi} \cos \alpha_e, \\C_{z_\xi} &= -C_{L_\xi} \cos \alpha_e - C_{D_\xi} \sin \alpha_e.\end{aligned}$$

The pitching-moment coefficient is unchanged by this rotation, so each C_{m_ξ} carries over directly. Thus the formulas retain possible drag and lift dependence on speed, pitch rate, angle-of-attack rate, and elevator deflection. No terms are removed by assuming small α_e , low Mach number, or negligible drag coupling.

Final Step: Calculate Dimensional Derivatives

Let $X_a = QSC_x$, $T = QSC_T$, and $Q_e = \frac{1}{2}\rho U_e^2$. Convert the nondimensional derivatives as follows:

1. Use $\hat{u} = u/U_e$ and $\partial Q/\partial u|_e = 2Q_e/U_e$:

$$X_u^a = \left. \frac{\partial(QSC_x)}{\partial u} \right|_e = \frac{Q_e S}{U_e} (C_{x_u} + 2C_{x_e}).$$

2. Apply the same chain rule to thrust:

$$T_u = \frac{Q_e S}{U_e} (C_{T_u} + 2C_{T_e}).$$

3. Add the aerodynamic and propulsive contributions:

$$X_u^{\text{total}} = \frac{Q_e S}{U_e} [C_{x_u} + C_{T_u} + 2(C_{x_e} + C_{T_e})].$$

4. Using $\Delta\alpha \simeq w/U_e$, with no thrust dependence on w ,

$$X_w^{\text{total}} = X_w^a = Q_e S C_{x_\alpha} \frac{\partial \alpha}{\partial w} = \frac{Q_e S}{U_e} C_{x_\alpha} = \frac{1}{2}\rho U_e S C_{x_\alpha}.$$

From Nondimensional to Dimensional: Longitudinal

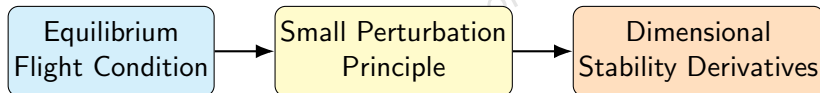
The axial column gives the derivative of the total axial force $X_{\text{total}} = X_a + T$; the Z and M columns remain aerodynamic for thrust along $\vec{x}_B$ through the center of mass. Assume

$$C_T = C_T(\hat{u}, \delta_p).$$

	X_{total} (N)	Z (N)	M (N·m)
u	$\frac{1}{2}\rho U_e S [C_{x_u} + C_{T_u} + 2(C_{x_e} + C_{T_e})]$	$\frac{1}{2}\rho U_e S (C_{z_u} + 2C_{z_e})$	$\frac{1}{2}\rho U_e S \bar{c} (C_{m_u} + 2C_{m_e})$
α	$\frac{1}{2}\rho U_e^2 S C_{x_\alpha}$	$\frac{1}{2}\rho U_e^2 S C_{z_\alpha}$	$\frac{1}{2}\rho U_e^2 S \bar{c} C_{m_\alpha}$
w	$\frac{1}{2}\rho U_e S C_{x_\alpha}$	$\frac{1}{2}\rho U_e S C_{z_\alpha}$	$\frac{1}{2}\rho U_e S \bar{c} C_{m_\alpha}$
q	$\frac{1}{4}\rho U_e S \bar{c} C_{x_q}$	$\frac{1}{4}\rho U_e S \bar{c} C_{z_q}$	$\frac{1}{4}\rho U_e S \bar{c}^2 C_{m_q}$
$\dot{\alpha}$	$\frac{1}{4}\rho U_e S \bar{c} C_{x_{\dot{\alpha}}}$	$\frac{1}{4}\rho U_e S \bar{c} C_{z_{\dot{\alpha}}}$	$\frac{1}{4}\rho U_e S \bar{c}^2 C_{m_{\dot{\alpha}}}$
δ_e	$\frac{1}{2}\rho U_e^2 S C_{x_{\delta_e}}$	$\frac{1}{2}\rho U_e^2 S C_{z_{\delta_e}}$	$\frac{1}{2}\rho U_e^2 S \bar{c} C_{m_{\delta_e}}$

C_{x_u}, C_{x_e} remain aerodynamic-only quantities; C_{T_u}, C_{T_e} are added explicitly to obtain the final X_u^{total} . $M_q, M_{\dot{\alpha}}$ carry an extra factor of $\bar{c}$ (vs. X, Z) from the moment arm. The w -row is the α -row divided by U_e , from $\alpha \approx w/U_e$ – use whichever row matches the chosen state vector.

Summary: Steps to Obtain Linearized Model



Case Study: [B747-100]

We now illustrate the development of the dynamic equations of motion by applying them to the Boeing 747-100 transport aircraft. The aircraft is cruising in horizontal flight at 40,000 ft and Mach 0.8 [**Etkin1996**]. Relevant data are provided below and in [**Heffley1972**].

$$\begin{aligned}W &= 2.83176 \times 10^6 \text{ N} \\I_{xx} &= 0.247 \times 10^8 \text{ kg } m^2 \\I_{yy} &= 0.449 \times 10^8 \text{ kg } m^2 \\I_{xz} &= -0.212 \times 10^7 \text{ kg } m^2 \\I_{zz} &= 6.73 \times 10^7 \text{ kg } m^2 \\U_e &= 235.9 \text{ m/s} \\\theta_e &= 0 \\C_{L_e} &= 0.654 \\C_{D_e} &= 0.0430\end{aligned}$$

The longitudinal dimensional derivatives are listed below. They include aerodynamic state/control derivatives and the propulsive control derivative X_{δ_p} :

ξ	X_ξ	Z_ξ	M_ξ
$u(m/s)$	-1.982×10^3	-2.595×10^4	1.593×10^4
$w(m/s)$	4.025×10^3	-9.030×10^4	-1.563×10^5
$q(rad/s)$	0	-4.524×10^5	-1.521×10^7
$\dot{w}(m/s^2)$	0	1.909×10^3	-1.702×10^4
$\delta_e (rad)$	-16.5299	-1.5794×10^6	-5.204×10^7
$\delta_p (-)$	8.4953×10^5	0	0

Units are the load unit divided by the row-variable unit; for example, X_u is in $N/(m/s)$, M_q is in $N \cdot m/(rad/s)$, and X_{δ_e} is in N/rad .

For comparison, the non-dimensional longitudinal derivative terms are listed below:

	C_x	C_z	C_m
$\hat{u}$	-0.1080	-0.1060	0.1043
α	0.2193	-4.92	-1.0230
$\hat{q}$	0	-5.9210	-23.92
$\hat{\alpha}$	0	5.8960	-6.3140
δ_e	-3.818×10^{-6}	-0.3648	-1.444
δ_p	0.1962	0	0

The lateral dimensional aerodynamic and control derivatives are listed below:

ξ	Y_ξ	L_ξ	N_ξ
$v(m/s)$	-1.610×10^4	-3.062×10^5	2.131×10^5
$p(rad/s)$	0	-1.076×10^7	-1.33×10^6
$r(rad/s)$	0	9.925×10^6	-8.934×10^6
$\delta_a (rad)$	0	-3.532×10^6	-5.095×10^4
$\delta_r (rad)$	4.962×10^5	1.801×10^6	-3.246×10^7

Units are the load unit divided by the row-variable unit.

For comparison, the non-dimensional lateral derivative terms are listed below:

	C_y	C_l	C_n
β	-0.8771	-0.2797	0.1946
$\hat{p}$	0	-0.3295	-0.0407
$\hat{r}$	0	0.3040	-0.2737
δ_a	0	-1.368×10^{-2}	-1.973×10^{-4}
δ_r	0.1146	6.976×10^{-3}	-0.1257

B747-100: Longitudinal State Matrix

Using $g = 9.81 \text{ m/s}^2$ gives

$$m = \frac{W}{g} = 2.8866 \times 10^5 \text{ kg.}$$

For $\underline{x}_{long} = [u \ w \ q \ \theta]^T$ and $\theta_e = 0$, we have $w_e = U_e \sin \theta_e = 0$.

The dimensional derivatives therefore give

$$\mathbf{A}_{long} = \begin{bmatrix} -0.006866 & 0.01394 & 0 & -9.810 \\ -0.09050 & -0.3149 & 235.893 & 0 \\ 0.0003891 & -0.003362 & -0.4282 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The matrix uses the state units [m/s, m/s, rad/s, rad].

B747-100: Lateral-Directional State Matrix

With

$$I'_{xx} = 4.0596 \times 10^{-8}, \quad I'_{zz} = 1.4899 \times 10^{-8}, \quad I'_{xz} = -1.2788 \times 10^{-9},$$

and $\underline{x}_{latr} = [v \ p \ r \ \phi]^T$, the dimensional derivatives give

$$\mathbf{A}_{latr} = \begin{bmatrix} -0.05577 & 0 & -235.900 & 9.810 \\ -0.01270 & -0.4351 & 0.4143 & 0 \\ 0.003567 & -0.006056 & -0.1458 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The matrix uses the state units [m/s, rad/s, rad/s, rad].

Summary

- 1 Rigid-Body Dynamics
- 2 Applied Flight Forces and Moments
- 3 Flight Dynamics: Linearization