

FUNDAMENTAL FLIGHT DYNAMICS & CONTROL

Lecture 01 Frame Definitions and Flight Kinematics

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1. Introduction

- ▶ **What is flight mechanics?** The study of an aircraft's motion under the aerodynamic, propulsive, and gravitational forces and moments acting on it – how it moves, and why.
- ▶ **How is it analyzed?** Model the aircraft as a rigid body governed by the Newton-Euler equations, build up aerodynamic/propulsive force and moment models, then linearize about an equilibrium (trim) flight condition to study its dynamic response – natural modes, stability, and handling qualities.
- ▶ **How is it controlled?** Design feedback laws, classical or modern, that use the aircraft's own control surfaces and throttle to shape that dynamic response – stabilizing what is naturally unstable and improving handling qualities.
- ▶ **How is trajectory control achieved?** Layer an outer guidance loop on top of the stabilized attitude/rate control to track a desired flight path, e.g. altitude-hold, heading-hold, waypoint navigation.

Applications of Flight Dynamics and Control

- ▶ Autopilot systems
- ▶ Unmanned aircraft



(a) Autopilot system



(b) Unmanned aircraft

Sources: MAPilots.org; U.S. Air Force photo by Master Sgt. Dennis J. Henry Jr. (public domain).

Applications of Flight Simulation

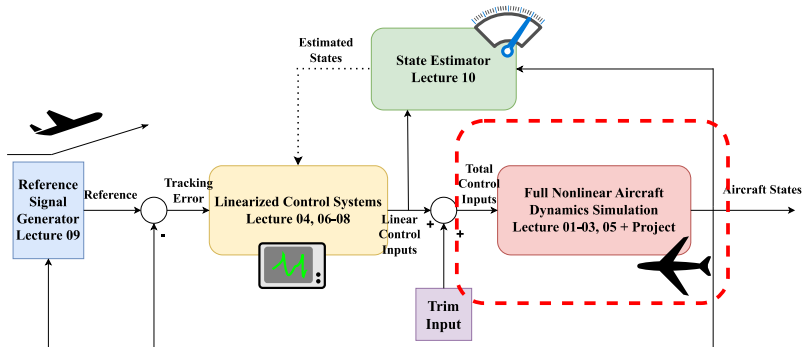


Figure 1: Full-flight simulator.

Source: CAE ©.

Course Learning Outcomes

- ▶ Describe aircraft motion using reference frames, rotation matrices, attitude coordinates, and rigid-body kinematics.
- ▶ Formulate nonlinear six-degree-of-freedom equations of motion from aerodynamic, propulsive, and gravitational forces and moments.
- ▶ Determine trim conditions and linearize the aircraft dynamics using aerodynamic stability and control derivatives.
- ▶ Analyze longitudinal and lateral-directional modes, stability, and time responses using MATLAB and Simulink.
- ▶ Design and assess feedback controllers for attitude, speed, yaw damping, and basic autopilot functions using classical and state-space methods.



2. Mathematical Preliminaries: Vectorial Kinematics

- ▶ vector representation: direction and magnitude, denoted by $\underline{\mathbf{v}}$;
- ▶ coordinate system: basis vectors, denoted by $\underline{\mathbf{e}}_i$, where $i = 1, 2, 3$;
- ▶ operations: dot product, cross product;
- ▶ norm (magnitude): $\|\underline{\mathbf{v}}\| \triangleq (\underline{\mathbf{v}} \cdot \underline{\mathbf{v}})^{1/2}$.

Given an ordered basis¹ $\{\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \underline{\mathbf{e}}_3\}$ for a 3-dimensional space, any vector $\underline{\mathbf{v}}$ can be expressed as a unique linear combination of the basis vectors:

$$\underline{\mathbf{v}} = v_1 \underline{\mathbf{e}}_1 + v_2 \underline{\mathbf{e}}_2 + v_3 \underline{\mathbf{e}}_3$$

- ▶ magnitude: determined by the coordinate values and basis geometry;
- ▶ direction: determined by the scaled basis vectors and their sum.

¹A set of linearly independent vectors that spans the entire space.

v_1, v_2, v_3 are the *coordinates of $\underline{\mathbf{v}}$ relative to the given ordered basis*. It is convenient to collect these coordinates in a column matrix, denoted by $\underline{\mathbf{v}}$ and also referred to as the *coordinate representation of $\underline{\mathbf{v}}$* :

$$\underline{\mathbf{v}} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \text{and} \quad \underline{\mathbf{v}}^T = [v_1 \ v_2 \ v_3]$$

For simplicity, $\underline{\mathbf{v}}$ denotes the coordinate representation of $\underline{\mathbf{v}}$, and $\underline{\mathbf{v}}^T$ denotes its transpose, i.e., a row vector.

For computation, the column $\underline{\mathbf{v}}$ is often called “the vector.” Vector operations can then be carried out on its coordinates using *matrix algebra*. This representation is meaningful only after the basis is fixed: the same $\underline{\mathbf{v}}$ has different coordinates in different frames.

Vector Representation in Matrix Form

A *vectrix* $\underline{\mathbf{F}}$ is a column matrix whose entries are the basis vectors:

$$\underline{\mathbf{F}} = \begin{bmatrix} \underline{\mathbf{e}}_1 \\ \underline{\mathbf{e}}_2 \\ \underline{\mathbf{e}}_3 \end{bmatrix} \quad \text{and} \quad \underline{\mathbf{F}}^T = [\underline{\mathbf{e}}_1 \quad \underline{\mathbf{e}}_2 \quad \underline{\mathbf{e}}_3]$$

Therefore, the vector $\underline{\mathbf{v}}$ can be written compactly as

$$\underline{\mathbf{v}} = \underline{\mathbf{F}}^T \underline{\mathbf{v}} = \underline{\mathbf{v}}^T \underline{\mathbf{F}} \quad (\text{vectrix form}) \quad (1)$$

Dot Product

scalar (dot) product: geometric viewpoint

$$\underline{\mathbf{u}} \cdot \underline{\mathbf{v}} \triangleq \|\underline{\mathbf{u}}\| \|\underline{\mathbf{v}}\| \cos \theta \quad (2)$$

where θ ($0 \leq \theta \leq \pi$) is the angle between the two vectors.

- ▶ $\underline{\mathbf{u}} \cdot \underline{\mathbf{v}} = 0$ when the two vectors are perpendicular, i.e., when $\theta = 90^\circ$;
- ▶ $\underline{\mathbf{v}} \cdot \underline{\mathbf{v}} = \|\underline{\mathbf{v}}\|^2 = v_1^2 + v_2^2 + v_3^2$

Denote the inner (scalar, dot) product of a pair of vectors $\underline{\underline{u}}$ and $\underline{\underline{v}}$ by

$$\begin{aligned}
 \underline{\underline{u}} \cdot \underline{\underline{v}} &= \underline{\underline{u}}^T \underline{\underline{F}} \cdot \underline{\underline{F}}^T \underline{\underline{v}} \\
 &= \underline{\underline{u}}^T \begin{bmatrix} \underline{\underline{e}}_1 \\ \underline{\underline{e}}_2 \\ \underline{\underline{e}}_3 \end{bmatrix} \cdot [\underline{\underline{e}}_1 \quad \underline{\underline{e}}_2 \quad \underline{\underline{e}}_3] \underline{\underline{v}} \\
 &= \underline{\underline{u}}^T \begin{bmatrix} \underline{\underline{e}}_1 \cdot \underline{\underline{e}}_1 & \underline{\underline{e}}_1 \cdot \underline{\underline{e}}_2 & \underline{\underline{e}}_1 \cdot \underline{\underline{e}}_3 \\ \underline{\underline{e}}_2 \cdot \underline{\underline{e}}_1 & \underline{\underline{e}}_2 \cdot \underline{\underline{e}}_2 & \underline{\underline{e}}_2 \cdot \underline{\underline{e}}_3 \\ \underline{\underline{e}}_3 \cdot \underline{\underline{e}}_1 & \underline{\underline{e}}_3 \cdot \underline{\underline{e}}_2 & \underline{\underline{e}}_3 \cdot \underline{\underline{e}}_3 \end{bmatrix} \underline{\underline{v}}
 \end{aligned} \tag{3}$$

Assume the basis vectors are *orthonormal*, that is

$$\underline{\mathbf{e}}_i \cdot \underline{\mathbf{e}}_j = 0 \quad (i \neq j)$$

$$\underline{\mathbf{e}}_i \cdot \underline{\mathbf{e}}_i = 1$$

leading to $\underline{\mathbf{F}} \cdot \underline{\mathbf{F}}^T = \mathbf{1}$, where $\mathbf{1}$ is the identity matrix. Therefore, the dot product of two vectors can be expressed using their coordinate vectors:

$$\underline{\mathbf{u}} \cdot \underline{\mathbf{v}} = \underline{\mathbf{u}}^T \underline{\mathbf{v}} = \underline{\mathbf{v}}^T \underline{\mathbf{u}} = \underline{\mathbf{v}} \cdot \underline{\mathbf{u}} \quad (4)$$

The dot product of a vector with itself defines its *norm* (to be exact, the 2-norm):

$$\|\underline{\mathbf{v}}\| \triangleq (\underline{\mathbf{v}} \cdot \underline{\mathbf{v}})^{1/2} \quad (5)$$

It is exactly the same as the norm of a numerical vector:

$$\|\underline{\mathbf{v}}\| \triangleq (\underline{\mathbf{v}}^T \underline{\mathbf{v}})^{1/2} = \|\underline{\mathbf{v}}\| \quad (6)$$

Cross Product

cross product: geometric viewpoint

Given vectors $\underline{u}$ and $\underline{v}$, the cross-product is defined as a vector $\underline{w}$, denoted by $\underline{w} = \underline{u} \times \underline{v}$ with magnitude

$$\|\underline{w}\| = \|\underline{u}\| \|\underline{v}\| \sin \theta \quad (7)$$

and direction perpendicular to both $\underline{u}$ and $\underline{v}$, chosen according to the right-hand rule.

Consider the cross product expressed in a *dextral, orthonormal* basis. The basis vectors satisfy

$$\underline{\mathbf{e}}_i \times \underline{\mathbf{e}}_i = \underline{\mathbf{0}} \quad (i = 1, 2, 3)$$

$$\underline{\mathbf{e}}_i \times \underline{\mathbf{e}}_j = \underline{\mathbf{e}}_k \quad (i : j : k = 1 : 2 : 3, 2 : 3 : 1, 3 : 1 : 2)$$

and the vector product is described as

$$\underline{\mathbf{u}} \times \underline{\mathbf{v}} = \underline{\mathbf{u}}^T \underline{\mathbf{F}} \times \underline{\mathbf{F}}^T \underline{\mathbf{v}} \quad (8)$$

$$\vec{u} \times \vec{v} = (u_1 \vec{e}_1 + u_2 \vec{e}_2 + u_3 \vec{e}_3) \times (v_1 \vec{e}_1 + v_2 \vec{e}_2 + v_3 \vec{e}_3)$$

$$= u_1 v_1 \underbrace{\vec{e}_1 \times \vec{e}_1}_{=\vec{0}} + u_1 v_2 \underbrace{\vec{e}_1 \times \vec{e}_2}_{=\vec{e}_3} + u_1 v_3 \underbrace{\vec{e}_1 \times \vec{e}_3}_{=-\vec{e}_2}$$

$$+ u_2 v_1 \underbrace{\vec{e}_2 \times \vec{e}_1}_{=-\vec{e}_3} + u_2 v_2 \underbrace{\vec{e}_2 \times \vec{e}_2}_{=\vec{0}} + u_2 v_3 \underbrace{\vec{e}_2 \times \vec{e}_3}_{=\vec{e}_1}$$

$$+ u_3 v_1 \underbrace{\vec{e}_3 \times \vec{e}_1}_{=\vec{e}_2} + u_3 v_2 \underbrace{\vec{e}_3 \times \vec{e}_2}_{=-\vec{e}_1} + u_3 v_3 \underbrace{\vec{e}_3 \times \vec{e}_3}_{=\vec{0}}$$

$$= \vec{F}^T \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix} = \vec{F}^T \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

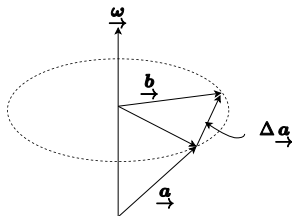
Therefore,

$$\underline{\mathbf{u}} \times \underline{\mathbf{v}} = \underline{\mathbf{F}}^T \underline{\mathbf{u}}^\times \underline{\mathbf{v}}, \text{ where } \underline{\mathbf{u}}^\times = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \quad (9)$$

Thus, $\underline{\mathbf{u}}^\times \underline{\mathbf{v}}$ is the coordinate representation of $\underline{\mathbf{u}} \times \underline{\mathbf{v}}$. In this course, $\underline{\mathbf{u}}^\times$ is called the *cross-product matrix* of $\underline{\mathbf{u}}$; it is skew-symmetric.

Geometric Meaning of Vector Operations

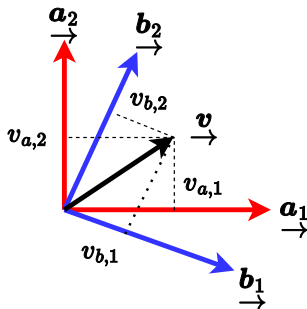
- ▶ dot product: measures alignment between two vectors; in particular, $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$.
- ▶ cross product: for a fixed-length vector $\vec{a}$ rotating with angular velocity $\vec{\omega}$, its rate of change is $\dot{\vec{a}} = \vec{\omega} \times \vec{a}$.



Frame Rotation and Rotation Matrix

A vector $\underline{v}$ has different coordinate representations in frames $\mathcal{F}_a$ and $\mathcal{F}_b$:

$$\underline{v} = \underline{F}_{\rightarrow a}^T \underline{v}_a = \underline{F}_{\rightarrow b}^T \underline{v}_b \quad (10)$$



The relationship between the two coordinate representations is determined by the relative orientation of the frames.

Consider two reference frames, $\mathcal{F}_a : \{\underline{\mathbf{a}}_1, \underline{\mathbf{a}}_2, \underline{\mathbf{a}}_3\}$ and $\mathcal{F}_b : \{\underline{\mathbf{b}}_1, \underline{\mathbf{b}}_2, \underline{\mathbf{b}}_3\}$. A rotation from $\mathcal{F}_a$ to $\mathcal{F}_b$ transforms the basis vectrix $\underline{\mathbf{F}}_a$ into $\underline{\mathbf{F}}_b$. Therefore,

$$\underline{\mathbf{v}} = \underline{\mathbf{F}}_a^T \underline{\mathbf{v}}_a = \underline{\mathbf{F}}_b^T \underline{\mathbf{v}}_b$$

$$\underline{\mathbf{v}}_b = \underline{\mathbf{F}}_b \cdot \underline{\mathbf{v}} = \underline{\mathbf{F}}_b \cdot \underline{\mathbf{F}}_a^T \underline{\mathbf{v}}_a \triangleq \underline{\mathbf{C}}_{ba} \underline{\mathbf{v}}_a$$

$$\underline{\mathbf{F}}_b \cdot \underline{\mathbf{F}}_a^T = \begin{bmatrix} \underline{\mathbf{b}}_1 \cdot \underline{\mathbf{a}}_1 & \underline{\mathbf{b}}_1 \cdot \underline{\mathbf{a}}_2 & \underline{\mathbf{b}}_1 \cdot \underline{\mathbf{a}}_3 \\ \underline{\mathbf{b}}_2 \cdot \underline{\mathbf{a}}_1 & \underline{\mathbf{b}}_2 \cdot \underline{\mathbf{a}}_2 & \underline{\mathbf{b}}_2 \cdot \underline{\mathbf{a}}_3 \\ \underline{\mathbf{b}}_3 \cdot \underline{\mathbf{a}}_1 & \underline{\mathbf{b}}_3 \cdot \underline{\mathbf{a}}_2 & \underline{\mathbf{b}}_3 \cdot \underline{\mathbf{a}}_3 \end{bmatrix} \triangleq \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \mathbf{C}_{ba}$$

The short expression then becomes

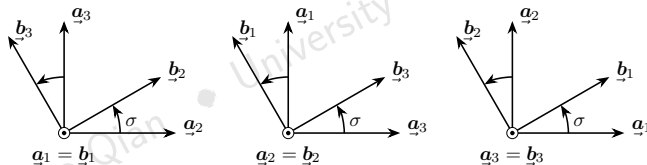
$$\underline{\mathbf{v}}_b = \mathbf{C}_{ba} \underline{\mathbf{v}}_a$$

$$\underline{\mathbf{F}}_b = \mathbf{C}_{ba} \underline{\mathbf{F}}_a$$

The rotation matrix $\mathbf{C}_{ba}$ is orthogonal, meaning that $\mathbf{C}_{ba}^T \mathbf{C}_{ba} = \mathbf{1}$, and its determinant is 1.

Principal Rotations

A frame may rotate about its first, second, or third principal axis. The corresponding rotation matrices are denoted by $\mathbf{C}_1(\sigma)$, $\mathbf{C}_2(\sigma)$, and $\mathbf{C}_3(\sigma)$, where σ is the rotation angle:



$\underline{F}_a \Rightarrow \underline{F}_b$ about $\underline{a}_1$ or $\underline{x}$

$$\begin{bmatrix} \underline{b}_1 \\ \underline{b}_2 \\ \underline{b}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \sigma & \sin \sigma \\ 0 & -\sin \sigma & \cos \sigma \end{bmatrix} \begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \\ \underline{a}_3 \end{bmatrix}$$
$$\underline{F}_b \triangleq \underline{C}_1(\sigma) \underline{F}_a \quad (11)$$

$\vec{F}_a \Rightarrow \vec{F}_b$ about $\vec{a}_2$ or $\underline{y}$

$$\begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} = \begin{bmatrix} \cos \sigma & 0 & -\sin \sigma \\ 0 & 1 & 0 \\ \sin \sigma & 0 & \cos \sigma \end{bmatrix} \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix}$$

$$\vec{F}_b \stackrel{\Delta}{=} C_2(\sigma) \vec{F}_a \quad (12)$$

$\vec{F}_a \Rightarrow \vec{F}_b$ about $\vec{a}_3$ or $\vec{z}$

$$\begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} = \begin{bmatrix} \cos \sigma & \sin \sigma & 0 \\ -\sin \sigma & \cos \sigma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix}$$

$$\vec{F}_b \stackrel{\Delta}{=} C_3(\sigma) \vec{F}_a \quad (13)$$

Nested Rotations

A rotation is rarely done in one step. Consider a *sequence* of rotations, from $\mathcal{F}_a$ to $\mathcal{F}_b$, and then from $\mathcal{F}_b$ to $\mathcal{F}_c$. Each step has its own rotation matrix,

$$\vec{F}_b = \mathbf{C}_{ba}\vec{F}_a \quad (14)$$

$$\vec{F}_c = \mathbf{C}_{cb}\vec{F}_b \quad (15)$$

Substituting the first relation into the second,

$$\vec{F}_c = \mathbf{C}_{cb}\mathbf{C}_{ba}\vec{F}_a \triangleq \mathbf{C}_{ca}\vec{F}_a \quad (16)$$

so the overall frame conversion is nothing but a *matrix product*

$$\mathbf{C}_{ca} = \mathbf{C}_{cb}\mathbf{C}_{ba} \quad (17)$$

Nested Rotations: A Sequence of n

The same product carries the column expression of any vector $\underline{v}$,

$$\underline{v}_c = \mathbf{C}_{cb}\underline{v}_b = \mathbf{C}_{cb}\mathbf{C}_{ba}\underline{v}_a \quad (18)$$

and the argument applies repeatedly. For n successive rotations $\mathcal{F}_0 \Rightarrow \mathcal{F}_1 \Rightarrow \cdots \Rightarrow \mathcal{F}_n$,

$$\mathbf{C}_{n0} = \mathbf{C}_{n,n-1} \cdots \mathbf{C}_{21}\mathbf{C}_{10} \quad (19)$$

- ▶ the matrices multiply *right to left*: the rotation applied first sits rightmost;
- ▶ matrix products do not commute, so the *order* of the rotations matters, $\mathbf{C}_{cb}\mathbf{C}_{ba} \neq \mathbf{C}_{ba}\mathbf{C}_{cb}$ in general.

Nested Rotations: Properties

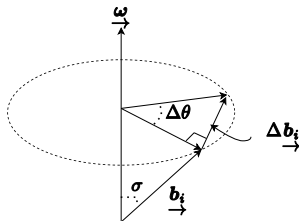
- ▶ a product of proper orthogonal matrices is proper orthogonal, so $\mathbf{C}_{n0}^T \mathbf{C}_{n0} = \mathbf{1}$ and $\det \mathbf{C}_{n0} = 1$: the result is again a valid rotation matrix;
- ▶ reversing the sequence transposes the product,
 $\mathbf{C}_{ac} = \mathbf{C}_{ca}^T = \mathbf{C}_{ba}^T \mathbf{C}_{cb}^T$.

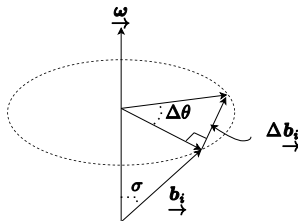
Consequently, any orientation can be assembled from three principal rotations, e.g., the aerospace 3-2-1 sequence

$$\mathbf{C}_{ba} = \mathbf{C}_1(\phi) \mathbf{C}_2(\theta) \mathbf{C}_3(\psi) \quad (20)$$

Time Derivative of Rotation Matrices

Assume that frame $\mathcal{F}_b$ rotates relative to frame $\mathcal{F}_a$ with angular velocity $\underline{\omega}$. Over a short time interval Δt , the basis vector $\underline{b}_i$ undergoes a small rotation $\Delta\theta = \|\underline{\omega}\|\Delta t$. Let σ denote the angle between $\underline{\omega}$ and $\underline{b}_i$.





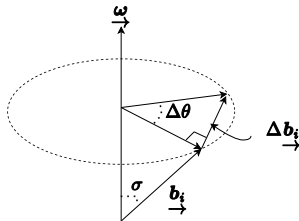
Note that $\|\underline{b}_i\| = 1$. The magnitude of $\Delta\underline{b}_i$ is given by:

$$\|\Delta\underline{b}_i\| \approx \|\underline{b}_i\| \sin(\sigma) \Delta\theta = \sin(\sigma) \Delta\theta = \|\underline{\omega} \times \underline{b}_i\| \Delta t$$

where $\Delta\theta = \|\underline{\omega}\| \Delta t$. The direction of $\Delta\underline{b}_i$ is perpendicular to the plane formed by $\underline{\omega}$ and $\underline{b}_i$, as determined by the right-hand rule.

Therefore,

$$\Delta\underline{b}_i = \underline{\omega} \times \underline{b}_i \Delta t$$



The time derivative of a basis vector of $\mathcal{F}_b$ relative to $\mathcal{F}_a$ is therefore

$$\dot{\vec{b}}_i = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{b}_i}{\Delta t} = \vec{\omega} \times \vec{b}_i$$

Express the angular velocity vector $\underline{\omega}$ using the basis vectors of frame $\mathcal{F}_b$:

$$\underline{\omega} = \omega_1 \underline{\mathbf{b}}_1 + \omega_2 \underline{\mathbf{b}}_2 + \omega_3 \underline{\mathbf{b}}_3 = \underline{\mathbf{F}}_b^T \underline{\omega}_b$$

$$\dot{\underline{\mathbf{b}}}_1 = \underline{\omega} \times \underline{\mathbf{b}}_1 = -\omega_2 \underline{\mathbf{b}}_3 + \omega_3 \underline{\mathbf{b}}_2$$

$$\dot{\underline{\mathbf{b}}}_2 = \underline{\omega} \times \underline{\mathbf{b}}_2 = -\omega_3 \underline{\mathbf{b}}_1 + \omega_1 \underline{\mathbf{b}}_3$$

$$\dot{\underline{\mathbf{b}}}_3 = \underline{\omega} \times \underline{\mathbf{b}}_3 = -\omega_1 \underline{\mathbf{b}}_2 + \omega_2 \underline{\mathbf{b}}_1$$

$$\dot{\underline{\mathbf{F}}}_b = -\underline{\omega}_b^\times \underline{\mathbf{F}}_b, \quad \underline{\omega}_b^\times = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Because the derivative is taken relative to $\mathcal{F}_a$, its basis vectors are fixed: $\dot{\underline{\mathbf{a}}}_i = \underline{\mathbf{0}}$. Hence, the time derivative of the rotation matrix

$\mathbf{C}_{ba}$ is

$$\dot{\mathbf{C}}_{ba} = \dot{\underline{\mathbf{F}}}_b \cdot \underline{\mathbf{F}}_a^T = -\underline{\omega}_b^\times \underline{\mathbf{F}}_b \cdot \underline{\mathbf{F}}_a^T = -\underline{\omega}_b^\times \mathbf{C}_{ba}$$

$$\dot{\mathbf{C}}_{ab} = \mathbf{C}_{ab} \underline{\omega}_b^\times$$

where the rotation matrix has the orthogonality property:

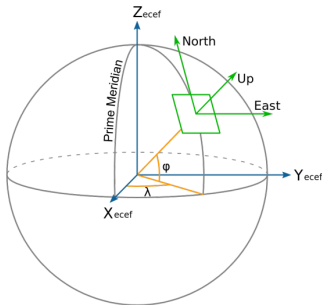
$$\begin{aligned}\mathbf{C}_{ab}^T \mathbf{C}_{ab} &= \mathbf{1} \\ \mathbf{C}_{ab}^T &= \mathbf{C}_{ba} = \mathbf{C}_{ab}^{-1} \\ \det \mathbf{C}_{ba} &= 1\end{aligned}\tag{21}$$

3. Frame Definitions

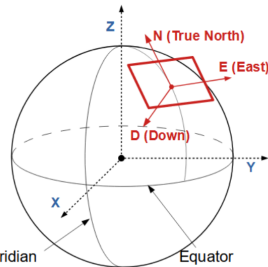
Key reference frames in flight dynamics:

- ▶ Earth-fixed frame $\mathcal{F}_E$: fixed to the Earth surface and treated as inertial in this course
- ▶ Body-fixed frame $\mathcal{F}_B$: rigid-body frame, fixed to the aircraft
- ▶ Wind frame $\mathcal{F}_W$: aerodynamic frame, aligned with the air-relative velocity
- ▶ Flight path frame $\mathcal{F}_P$: trajectory frame, aligned with the velocity relative to the Earth (ground speed)

ENU vs NED convention



(a) ENU (East-North-Up) frame² (b) NED (North-East-Down) frame³



²Source: https://upload.wikimedia.org/wikipedia/commons/thumb/7/73/ECEF_ENU_Longitude_Latitude_relationships.svg/1024px-ECEF_ENU_Longitude_Latitude_relationships.svg.png

³Source: https://support.sbg-systems.com/sc/kb/files/latest/11242072/11242328/1/1613149262462/NED_schema.PNG

The Earth-fixed frame $\mathcal{F}_E$

The Earth-surface-fixed frame $\mathcal{F}_E : \{O_E, \underline{x}_E, \underline{y}_E, \underline{z}_E\}$ has its origin O_E at a fixed location on the Earth's surface and adopts the *NED* convention:

- ▶ $\underline{x}_E$ points *north*;
- ▶ $\underline{y}_E$ points *east*;
- ▶ $\underline{z}_E$ points *down*, vertically into the Earth surface,

so that $\{\underline{x}_E, \underline{y}_E, \underline{z}_E\}$ is a dextral, orthonormal triad.

Throughout this course $\mathcal{F}_E$ is **assumed to be an inertial frame**: the equations of motion are written directly in $\mathcal{F}_E$, and the effects of the Earth's rotation and curvature are ignored.

The Body-Fixed Frame $\mathcal{F}_B$

The body-fixed reference frame $\mathcal{F}_B : \{\underline{O}_{cm}, \underline{x}_B, \underline{y}_B, \underline{z}_B\}$ is fixed to the vehicle, so the angular velocity of the frame is the same as that of the body. Its origin O_{cm} is usually chosen at the centre of mass, and it adopts the *FRD* convention:

- ▶ $\underline{x}_B$ points *forward*, towards the nose of the aircraft;
- ▶ $\underline{y}_B$ points *right*, towards the right-wing tip;
- ▶ $\underline{z}_B$ points *down*, perpendicular to $\underline{x}_B$,

so that $\{\underline{x}_B, \underline{y}_B, \underline{z}_B\}$ is a dextral, orthonormal triad. The axes $\underline{x}_B$ and $\underline{z}_B$ lie in the plane of vertical symmetry, and $\underline{y}_B$ is normal to it.

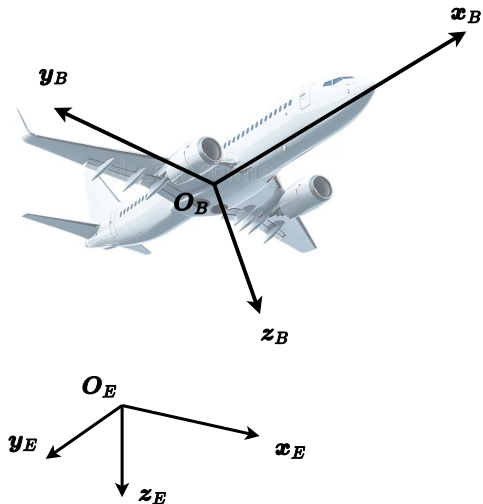
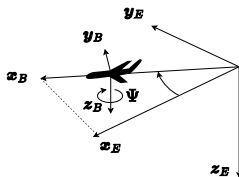


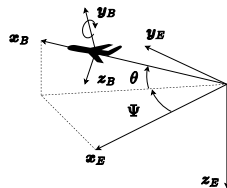
Figure 2: The body-fixed frame $\mathcal{F}_B$.

Link Between $\mathcal{F}_B$ and $\mathcal{F}_E$: Aircraft Euler Angles

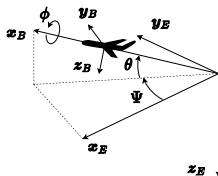
The Euler angles (ψ, θ, ϕ) rotate the Earth frame $\mathcal{F}_E$ into the body frame $\mathcal{F}_B$ through yaw, pitch, and roll, respectively. The rotations are performed in the following order:



Step 1: Yaw



Step 2: Pitch



Step 3: Roll

1. a rotation of ψ about $\underline{z}_E = \underline{z}_1$, where ψ is called the *azimuth angle* or *yaw angle*: $-\pi \leq \psi < \pi$ or $0 \leq \psi < 2\pi$,

$$\{\underline{x}_E, \underline{y}_E, \underline{z}_E\} \implies \{\underline{x}_1, \underline{y}_1, \underline{z}_1\}$$

$$\begin{bmatrix} \underline{x}_1 \\ \underline{y}_1 \\ \underline{z}_1 \end{bmatrix} = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{x}_E \\ \underline{y}_E \\ \underline{z}_E \end{bmatrix}, \quad \mathbf{C}_3(\psi) \quad (22)$$

2. a rotation of θ about $\underline{y}_1 = \underline{y}_2$, where θ is called the *elevation angle* or *pitch angle*: $-\pi/2 < \theta < \pi/2$,

$$\{\underline{x}_1, \underline{y}_1, \underline{z}_1\} \implies \{\underline{x}_2, \underline{y}_2, \underline{z}_2\}$$

$$\begin{bmatrix} \underline{x}_2 \\ \underline{y}_2 \\ \underline{z}_2 \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{y}_1 \\ \underline{z}_1 \end{bmatrix}, \quad \mathbf{C}_2(\theta) \quad (23)$$

3. a rotation of ϕ about $\underline{x}_2 = \underline{x}_B$, where ϕ is called the *roll angle*: $-\pi \leq \phi < \pi$ or $0 \leq \phi < 2\pi$,

$$\{\underline{x}_2, \underline{y}_2, \underline{z}_2\} \Rightarrow \{\underline{x}_B, \underline{y}_B, \underline{z}_B\}$$

$$\begin{bmatrix} \underline{x}_B \\ \underline{y}_B \\ \underline{z}_B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} \underline{x}_2 \\ \underline{y}_2 \\ \underline{z}_2 \end{bmatrix}, \quad \mathbf{C}_1(\phi) \quad (24)$$

The rotation matrix from $\mathcal{F}_E$ to $\mathcal{F}_B$ is given by

$$\underline{\mathbf{F}}_B = \mathbf{C}_{BE} \underline{\mathbf{F}}_E \quad (25)$$

where $\mathbf{C}_{BE} = \mathbf{C}_1(\phi)\mathbf{C}_2(\theta)\mathbf{C}_3(\psi) =$

$$\begin{bmatrix} \cos \theta \cos \psi & \cos \theta \sin \psi & -\sin \theta \\ \sin \phi \sin \theta \cos \psi - \cos \phi \sin \psi & \sin \phi \sin \theta \sin \psi + \cos \phi \cos \psi & \sin \phi \cos \theta \\ \cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi & \cos \phi \sin \theta \sin \psi - \sin \phi \cos \psi & \cos \phi \cos \theta \end{bmatrix} \quad (26)$$

and its inverse $\mathbf{C}_{EB} = \mathbf{C}_{BE}^T =$

$$\begin{bmatrix} \cos \theta \cos \psi & \sin \phi \sin \theta \cos \psi - \cos \phi \sin \psi & \cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi \\ \cos \theta \sin \psi & \sin \phi \sin \theta \sin \psi + \cos \phi \cos \psi & \cos \phi \sin \theta \sin \psi - \sin \phi \cos \psi \\ -\sin \theta & \sin \phi \cos \theta & \cos \phi \cos \theta \end{bmatrix} \quad (27)$$

Where Lift and Drag are Defined: Wind Frame $\mathcal{F}_W$

The wind frame $\mathcal{F}_W : \{O_{cm}, \underline{x}_W, \underline{y}_W, \underline{z}_W\}$ has its origin fixed to the vehicle, usually at the centre of mass. Its axis $\underline{x}_W$ is directed along the velocity vector $\underline{v}$ of the vehicle relative to the *atmosphere* (airspeed), and $\underline{z}_W$ remains in the plane of symmetry. This is what makes $\mathcal{F}_W$ the natural frame for the aerodynamic forces: *drag* is by definition opposite to the airspeed, and *lift* normal to it, so both act along the *negative* wind axes,

$$\underline{D} = -D \underline{x}_W, \quad \underline{L} = -L \underline{z}_W \quad (28)$$

The orientation of $\mathcal{F}_B$ relative to $\mathcal{F}_W$ is described by the aerodynamic angles α (angle of attack) and β (sideslip angle):

$$\mathbf{C}_{BW} = \mathbf{C}_1(0)\mathbf{C}_2(\alpha)\mathbf{C}_3(-\beta) \quad (29)$$

Calculate Aerodynamic Angles

The aircraft velocity vector $\underline{v}$ and its coordinates $\underline{v}_B$ are given by

$$\underline{v} = \underline{F}_B^T \underline{v}_B \quad \underline{v}_B = \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \quad V = \|\underline{v}_B\| \quad (30)$$

Angle of Attack:

$$\alpha = \text{atan2}(w, u), \quad -\pi < \alpha \leq \pi \quad (31)$$

Sideslip Angle:

$$\beta = \sin^{-1} \frac{v}{V}, \quad -\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}, \quad V > 0 \quad (32)$$

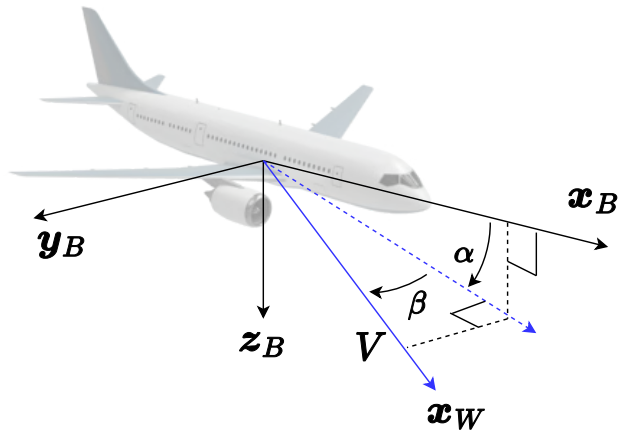


Figure 3: Aerodynamic angles.

Link Between $\mathcal{F}_W$ and $\mathcal{F}_B$

The rotation matrix from $\mathcal{F}_W$ to $\mathcal{F}_B$ is obtained by the following Euler rotation sequence: $(\psi, \theta, \phi) = (-\beta, \alpha, 0)$, i.e.

$$\underline{\mathbf{F}}_B = \mathbf{C}_{BW} \underline{\mathbf{F}}_W \quad (33)$$

where

$$\begin{aligned} \mathbf{C}_{BW} &= \mathbf{C}_1(0) \mathbf{C}_2(\alpha) \mathbf{C}_3(-\beta) \\ &= \begin{bmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \beta & -\sin \beta & 0 \\ \sin \beta & \cos \beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos \alpha \cos \beta & -\cos \alpha \sin \beta & -\sin \alpha \\ \sin \beta & \cos \beta & 0 \\ \sin \alpha \cos \beta & -\sin \alpha \sin \beta & \cos \alpha \end{bmatrix} \end{aligned} \quad (34)$$

Flight Path Frame $\mathcal{F}_P$

The flight path frame $\mathcal{F}_P : \{O_{cm}, \underline{x}_P, \underline{y}_P, \underline{z}_P\}$ has $\underline{x}_P$ directed along the velocity vector of the vehicle relative to the *Earth* (ground speed), so O_{cm} traces out the trajectory of the vehicle and $\underline{x}_P$ is always tangent to it. $\mathcal{F}_P$ is reached directly from $\mathcal{F}_E$ by the heading angle σ , flight path angle γ , and bank angle μ :

$$\mathbf{C}_{PE} = \mathbf{C}_1(\mu)\mathbf{C}_2(\gamma)\mathbf{C}_3(\sigma) \quad (35)$$

If the atmosphere is at rest, ground speed equals airspeed, and $\mathcal{F}_P \equiv \mathcal{F}_W$.

The rotation matrix $\mathbf{C}_{EP} = \mathbf{C}_{PE}^T$ is

$$\mathbf{C}_{EP} = \begin{bmatrix} \cos \gamma \cos \sigma & \sin \mu \sin \gamma \cos \sigma - \cos \mu \sin \sigma & \cos \mu \sin \gamma \cos \sigma + \sin \mu \sin \sigma \\ \cos \gamma \sin \sigma & \sin \mu \sin \gamma \sin \sigma + \cos \mu \cos \sigma & \cos \mu \sin \gamma \sin \sigma - \sin \mu \cos \sigma \\ -\sin \gamma & \sin \mu \cos \gamma & \cos \mu \cos \gamma \end{bmatrix} \quad (36)$$

Conversion between $\mathcal{F}_P$ and $\mathcal{F}_W$

Assuming still air ($\mathcal{F}_P \equiv \mathcal{F}_W$), both constructions describe the same rotation from $\mathcal{F}_E$:

$$\mathbf{C}_1(\mu)\mathbf{C}_2(\gamma)\mathbf{C}_3(\sigma) = \mathbf{C}_{WE} = \mathbf{C}_3(\beta)\mathbf{C}_2(-\alpha)\mathbf{C}_1(\phi)\mathbf{C}_2(\theta)\mathbf{C}_3(\psi)$$

Equating entries gives (μ, γ, σ) explicitly in terms of $(\alpha, \beta, \phi, \theta, \psi)$:

$$\sin \gamma = \cos \alpha \cos \beta \sin \theta - \sin \alpha \cos \beta \cos \phi \cos \theta - \sin \beta \sin \phi \cos \theta$$

$$\tan \mu = \frac{\sin \phi \cos \beta \cos \theta + \sin \beta (\sin \theta \cos \alpha - \sin \alpha \cos \phi \cos \theta)}{\sin \alpha \sin \theta + \cos \alpha \cos \phi \cos \theta}$$

Conversion between $\mathcal{F}_P$ and $\mathcal{F}_W$ (cont'd)

$$\sigma = \psi + \text{atan2}(A, B)$$

$$A = \sin \beta \cos \phi - \sin \alpha \sin \phi \cos \beta$$

$$B = \cos \alpha \cos \beta \cos \theta + \sin \alpha \cos \beta \cos \phi \sin \theta + \sin \beta \sin \phi \sin \theta$$

As a check, when $\alpha = \beta = 0$ (no aerodynamic angles), $\mathcal{F}_W \equiv \mathcal{F}_B$ and the formulas reduce to $\gamma = \theta$, $\mu = \phi$, and $\sigma = \psi$, as expected. In general, however, $\mu \neq \phi$ and $\gamma \neq \theta$; their relationships depend on the complete aircraft orientation and aerodynamic angles.

4. Aircraft Kinematics

The principal vectors and matrices used in aircraft kinematics are defined as follows:

- ▶ inertial position: $\underline{x}_E$
- ▶ inertial- and body-frame velocities: $\underline{v}_E$ and $\underline{v}_B$
- ▶ angular velocity: $\underline{\omega}_B$
- ▶ rotation matrices: $\mathbf{C}_{EB}$ and $\mathbf{C}_{BE} = \mathbf{C}_{EB}^T$

Aircraft Translational Motion

Aircraft inertial position:

$$\underline{\mathbf{x}}_E = \begin{bmatrix} x_E \\ y_E \\ z_E \end{bmatrix}$$

Aircraft velocity vector $\underline{\mathbf{v}}$

$$\underline{\mathbf{v}} = \underline{\mathbf{F}}_E^T \underline{\mathbf{v}}_E$$

$$\underline{\mathbf{v}}_E = \begin{bmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{bmatrix} = \dot{\underline{\mathbf{x}}}_E$$

$$\underline{\mathbf{v}} = \underline{\mathbf{F}}_B^T \underline{\mathbf{v}}_B$$

$$\underline{\mathbf{v}}_B = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \neq \dot{\underline{\mathbf{x}}}_E$$

$$\underline{\mathbf{v}}_E = \underline{\mathbf{C}}_{EB} \underline{\mathbf{v}}_B$$

Aircraft Rotational Motion

The angular velocity of $\mathcal{F}_B$ relative to $\mathcal{F}_E$ is denoted by $\underline{\omega}$. Its coordinate representation in $\mathcal{F}_B$ is

$$\underline{\omega} = \mathbf{F}_B^T \underline{\omega}_B$$

where

$$\underline{\omega}_B = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

The time-derivative of the rotation matrix $\mathbf{C}_{EB}$ is

$$\dot{\mathbf{C}}_{EB} = \mathbf{C}_{EB} \underline{\omega}_B^\times$$

Time Derivative of Euler Angles

If Euler angles (ψ, θ, ϕ) describe the aircraft attitude, their time derivatives can be calculated from $\underline{\omega}_B$. The angular velocity associated with the Euler rotation sequence is

$$\underline{\omega} = \dot{\psi} \underline{z}_1 + \dot{\theta} \underline{y}_2 + \dot{\phi} \underline{x}_B$$

$$\begin{bmatrix} \underline{x}_B & \underline{y}_2 & \underline{z}_1 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \underline{x}_B & \underline{y}_2 & \underline{z}_1 \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

Thus, $\underline{\omega}$ is represented using the successive rotation axes $\underline{z}_1$, $\underline{y}_2$, and $\underline{x}_B$.

Recall the rotation matrix about a principal axis:

$$\begin{bmatrix} \underline{x}_B \\ \underline{y}_B \\ \underline{z}_B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} \underline{x}_2 \\ \underline{y}_2 \\ \underline{z}_2 \end{bmatrix}, \quad \mathbf{C}_1(\phi) \quad (37)$$

The inverse relation is

$$\begin{bmatrix} \underline{x}_2 \\ \underline{y}_2 \\ \underline{z}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} \underline{x}_B \\ \underline{y}_B \\ \underline{z}_B \end{bmatrix}$$

Hence,

$$\underline{y}_2 = \cos \phi \underline{y}_B - \sin \phi \underline{z}_B = \begin{bmatrix} \underline{x}_B & \underline{y}_B & \underline{z}_B \end{bmatrix} \begin{bmatrix} 0 \\ \cos \phi \\ -\sin \phi \end{bmatrix}$$

In addition,

$$\vec{x}_2 = \vec{x}_B, \quad \vec{z}_2 = \sin \phi \vec{y}_B + \cos \phi \vec{z}_B$$

$$\begin{bmatrix} \vec{x}_2 \\ \vec{y}_2 \\ \vec{z}_2 \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} \vec{x}_1 \\ \vec{y}_1 \\ \vec{z}_1 \end{bmatrix}, \quad \mathbf{C}_2(\theta) \quad (38)$$

$$\begin{bmatrix} \vec{x}_1 \\ \vec{y}_1 \\ \vec{z}_1 \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} \vec{x}_2 \\ \vec{y}_2 \\ \vec{z}_2 \end{bmatrix}$$

$$\begin{bmatrix} \underline{x}_1 \\ \underline{y}_1 \\ \underline{z}_1 \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} \underline{x}_2 \\ \underline{y}_2 \\ \underline{z}_2 \end{bmatrix}$$

$$\begin{aligned} \underline{z}_1 &= -\sin \theta \underline{x}_2 + \cos \theta \underline{z}_2 \\ &= -\sin \theta \underline{x}_B + \cos \theta (\sin \phi \underline{y}_B + \cos \phi \underline{z}_B) \\ &= [\underline{x}_B \quad \underline{y}_B \quad \underline{z}_B] \begin{bmatrix} -\sin \theta \\ \cos \theta \sin \phi \\ \cos \theta \cos \phi \end{bmatrix} \end{aligned}$$

$$[\underline{x}_B \ \underline{y}_2 \ \underline{z}_1] = [\underline{x}_B \ \underline{y}_B \ \underline{z}_B] \underbrace{\begin{bmatrix} 1 & 0 & -\sin \theta \\ 0 & \cos \phi & \cos \theta \sin \phi \\ 0 & -\sin \phi & \cos \theta \cos \phi \end{bmatrix}}_{\text{coordinate transformation matrix}}$$

$$\begin{aligned} [\underline{x}_B \ \underline{y}_B \ \underline{z}_B] \begin{bmatrix} p \\ q \\ r \end{bmatrix} &= [\underline{x}_B \ \underline{y}_2 \ \underline{z}_1] \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \\ &= [\underline{x}_B \ \underline{y}_B \ \underline{z}_B] \begin{bmatrix} 1 & 0 & -\sin \theta \\ 0 & \cos \phi & \cos \theta \sin \phi \\ 0 & -\sin \phi & \cos \theta \cos \phi \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \end{aligned}$$

Therefore, the Euler angles can be updated directly from the angular velocity:

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin \theta \\ 0 & \cos \phi & \sin \phi \cos \theta \\ 0 & -\sin \phi & \cos \phi \cos \theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \triangleq \mathbf{S}_B \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \quad (39)$$

To update attitude, one can use either of the following:

- ▶ Use rotation matrix: $\dot{\mathbf{C}}_{EB} = \mathbf{C}_{EB} \underline{\boldsymbol{\omega}}_B^x$;
- ▶ Use Euler angles:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \mathbf{S}_B^{-1} \underline{\boldsymbol{\omega}}_B = \begin{bmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi \sec \theta & \cos \phi \sec \theta \end{bmatrix} \underline{\boldsymbol{\omega}}_B.$$

The Euler-angle form is singular when $\cos \theta = 0$ (that is, at $\theta = \pm\pi/2$), where $\tan \theta$ and $\sec \theta$ are undefined. This is the 3-2-1 Euler-angle gimbal lock.

Self-study: derive the closed-form inverse of $\mathbf{S}_B$ using reference deck R1.

Aircraft Velocity Expressions Review

The aircraft velocity vector $\underline{v}$ has different coordinate representations in different reference frames:

$$\underline{v}_E = \begin{bmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{bmatrix} = \mathbf{C}_{EB} \underline{v}_B = \mathbf{C}_{EB} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \quad (40)$$

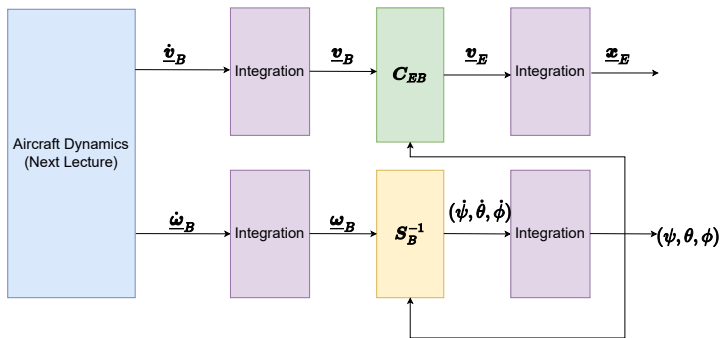
$$\underline{v}_E = \begin{bmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{bmatrix} = \mathbf{C}_{EP} \begin{bmatrix} V \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} V \cos \gamma \cos \sigma \\ V \cos \gamma \sin \sigma \\ -V \sin \gamma \end{bmatrix} \quad (41)$$

$$\underline{v}_B = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{C}_{BW} \begin{bmatrix} V \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} V \cos \alpha \cos \beta \\ V \sin \beta \\ V \sin \alpha \cos \beta \end{bmatrix} \quad (42)$$

Here γ and σ are the flight path angles, so the second line uses $\mathbf{C}_{EP}$; in still air $\mathcal{F}_P \equiv \mathcal{F}_W$ and the two speeds coincide.

Aircraft Kinematics Block Diagram

The aircraft kinematics block diagram is shown below.



Summary

- 1 Introduction
- 2 Mathematical Preliminaries: Vectorial Kinematics
- 3 Frame Definitions
- 4 Aircraft Kinematics