

AER1202 Tutorial 1

Rotation Matrices, Rotating-Frame Derivatives, and `fsolve`

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1. Tutorial Roadmap

1. Compose principal-axis rotation matrices in the correct order.
2. Differentiate coordinate expressions between rotating frames.
3. Formulate and check a two-equation nonlinear solve using MATLAB `fsolve`.

2. Rotation Matrix

Using the course convention, consider

- ▶ first, a rotation through α about axis 3;
- ▶ second, a rotation through β about the new axis 1.

$$\mathbf{C}_3(\alpha) = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{C}_1(\beta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & \sin \beta \\ 0 & -\sin \beta & \cos \beta \end{bmatrix}.$$

Then, what will the overall rotation matrix be?

2. Rotation Matrix

The rightmost matrix acts first:

$$\mathbf{C} = \mathbf{C}_1(\beta)\mathbf{C}_3(\alpha).$$

Quick check

The adjacent frame labels must cancel in a matrix product, just like units, i.e. the shapes of matrices must match

Multiplying row by column gives

$$\mathbf{C} = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\cos \beta \sin \alpha & \cos \beta \cos \alpha & \sin \beta \\ \sin \beta \sin \alpha & -\sin \beta \cos \alpha & \cos \beta \end{bmatrix}.$$

2. Rotation Matrix

Things to check:

- If setting $\beta = 0$, what will happen to the result?

$$\mathbf{C} = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\cos 0 \sin \alpha & \cos 0 \cos \alpha & \sin 0 \\ \sin 0 \sin \alpha & -\sin 0 \cos \alpha & \cos 0 \end{bmatrix} = ?$$

2. Rotation Matrix

Recall the properties of rotation matrices:

► $\mathbf{C}^T = \mathbf{C}^{-1}$, i.e. $\mathbf{C}^T \mathbf{C} = \mathbf{I}$

► $\det \mathbf{C} = +1$

Please verify using the calculated $\mathbf{C}$ in previous slides

Hint:

$$\det(\mathbf{A}) = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

2. Rotation Matrix

What will happen if we reverse the factors in a rotation matrix?

- ▶ For example:

$$\mathbf{C} = \mathbf{C}_3(\alpha)\mathbf{C}_1(\beta).$$

- ▶ What rotation does it describe?
- ▶ Why does it happen?

3. Rotating-frame Derivatives

Let the coordinate columns of the same quantity be related by

$$\underline{\mathbf{a}}_E(t) = \mathbf{C}_{EB}(t)\underline{\mathbf{a}}_B(t).$$

Both factors may depend on time. Apply the matrix product rule:

$$\dot{\underline{\mathbf{a}}}_E = \dot{\mathbf{C}}_{EB}\underline{\mathbf{a}}_B + \mathbf{C}_{EB}\dot{\underline{\mathbf{a}}}_B.$$

Recall the result from the lecture:

$$\dot{\mathbf{C}}_{EB} = \mathbf{C}_{EB}\underline{\boldsymbol{\omega}}_B^\times,$$

the coordinate derivative becomes

$$\dot{\underline{\mathbf{a}}}_E = \mathbf{C}_{EB} \left(\dot{\underline{\mathbf{a}}}_B + \underline{\boldsymbol{\omega}}_B^\times \underline{\mathbf{a}}_B \right).$$

3. Rotating-frame Derivatives

If a coordinate column depends on a scalar angle $\chi(t)$, then

$$\dot{\underline{a}}_B = \frac{\partial \underline{a}_B}{\partial \chi} \dot{\chi}.$$

For example,

$$\frac{d}{dt} \cos \chi(t) = -\sin \chi(t) \dot{\chi}, \quad \frac{d}{dt} \sin \chi(t) = \cos \chi(t) \dot{\chi}.$$

Two distinct sources of change

- ▶ $\dot{\underline{a}}_B$: the coordinates change within $\mathcal{F}_B$;
- ▶ $\dot{\mathbf{C}}_{EB} \underline{a}_B$: the frame orientation changes relative to $\mathcal{F}_E$.

Do not discard either term until its time dependence has been checked.

3. Rotating-frame Derivatives

Example: velocity of a point on a propeller

Let a propeller spin about x_B . A blade point at radius R has body-frame coordinates

$$\underline{\mathbf{r}}_B(t) = \begin{bmatrix} 0 \\ R \cos \chi(t) \\ R \sin \chi(t) \end{bmatrix}, \quad \dot{\chi} = \Omega.$$

The chain rule gives its coordinate rate relative to the hub:

$$\dot{\underline{\mathbf{r}}}_B = \begin{bmatrix} 0 \\ -R\Omega \sin \chi \\ R\Omega \cos \chi \end{bmatrix}.$$

If the aircraft body also rotates, convert the relative offset to $\mathcal{F}_E$ and differentiate:

$$\dot{\underline{\mathbf{r}}}_E = \mathbf{C}_{EB} \left(\dot{\underline{\mathbf{r}}}_B + \underline{\boldsymbol{\omega}}_B^{\times} \underline{\mathbf{r}}_B \right).$$

What will happen if any factor is fixed in the corresponding frame?

3. Rotating-frame Derivatives

Checks:

- ▶ With fixed aircraft attitude, set $\underline{\omega}_B = \underline{0}$.
- ▶ With a locked propeller, set $\Omega = 0$. The remaining term should match the physical case.

4. Nonlinear Equations with fsolve

Example: find (x, y) satisfying

$$x^2 + y^2 = 5, \quad x - y = 1.$$

Write the problem in residual form:

$$\vec{F}(\vec{z}) = \begin{bmatrix} x^2 + y^2 - 5 \\ x - y - 1 \end{bmatrix} = \vec{0}, \quad \vec{z} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Before running MATLAB

Confirm that the residual has two entries, the initial guess has two entries, and each residual is dimensionally consistent. A sketch shows that more than one root may exist, so the initial guess matters.

4. Nonlinear Equations with fsolve

```
fun = @(z) [z(1)^2 + z(2)^2 - 5;  
            z(1) - z(2) - 1];  
  
z0 = [1.5; 0.5];  
opts = optimoptions('fsolve', 'Display', 'iter');  
  
[z,fval,exitflag] = fsolve(fun,z0,opts);  
  
residual_norm = norm(fval,inf)
```

4. Nonlinear Equations with fsolve

Checks after the solve

- ▶ Require `exitflag > 0`; convergence text alone is not enough.
- ▶ Confirm `residual_norm` is small.
- ▶ Substitute the returned values into both original equations.
- ▶ Try a substantially different initial guess and compare the root.

5. Key Takeaways

1. Order of factors in a rotation matrix, and the reason
2. What two terms result from differentiating $\underline{\mathbf{a}}_E = \mathbf{C}_{EB}\underline{\mathbf{a}}_B$?
3. Why can constant body-frame coordinates have a nonzero inertial rate?
4. What three checks establish that an fsolve result is credible?