

Aerodynamic Loads and Stability Derivatives

From Nonlinear Simulation to Linearized Models

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1. Aerodynamic Loads for Nonlinear Simulation

Given the dynamic pressure $Q = \frac{1}{2}\rho V^2$ and reference geometry S (area), b (span), $\bar{c}$ (mean chord), every aerodynamic force and moment is the corresponding coefficient scaled back up by Q , S , and – for moments – one reference length:

Quantity	Formula	Reference
Lift	$L = Q S C_L$	–
Drag	$D = Q S C_D$	–
Side force	$Y = Q S C_Y$	–
Rolling moment	$L = Q S b C_l$	b (span)
Pitching moment	$M = Q S \bar{c} C_m$	$\bar{c}$ (mean chord)
Yawing moment	$N = Q S b C_n$	b (span)

(L_a denotes aerodynamic lift; bare L is the *rolling* moment – the two are unrelated despite the shared symbol.)

Normalized Angular and AOA Rates

The stability derivatives C_{m_q} , $C_{m_{\dot{\alpha}}}$, C_{l_p} , C_{n_r} , etc. are defined with respect to *normalized* (nondimensional) rates, not the raw rates in rad/s – this is what makes them dimensionless and comparable across aircraft of different size and speed.

Longitudinal rates use the mean aerodynamic chord $\bar{c}$ as the reference length:

Rate	Dimensional	Normalized	Reference length
Pitch rate	q	$\hat{q} = \frac{q\bar{c}}{2U_e}$	$\bar{c}$ (mean chord)
AOA rate	$\dot{\alpha}$	$\hat{\alpha} = \frac{\dot{\alpha}\bar{c}}{2U_e}$	$\bar{c}$

Normalized Angular and AOA Rates (cont'd)

Lateral-directional rates use the wingspan b instead – both cases via the same time scale, $\bar{c}/2U_e$ or $b/2U_e$ respectively:

Rate	Dimensional	Normalized	Reference length
Roll rate	p	$\hat{p} = \frac{pb}{2U_e}$	b (span)
Yaw rate	r	$\hat{r} = \frac{rb}{2U_e}$	b

This is exactly the extra $\bar{c}/2U_e$ or $b/2U_e$ factor that shows up in the dimensionalization formulas next.

A Note on U_e : Linear vs. Nonlinear Simulation

- ▶ **Linearized (small-perturbation) model:** U_e is a fixed constant – the one trim airspeed the model was linearized about. The stability derivatives were measured/computed at that condition, and the whole framework assumes $V(t) \approx U_e$.
- ▶ **Full nonlinear 6DOF simulation:** replace U_e with the **instantaneous airspeed** $V(t)$ everywhere – both in the dynamic pressure and in the rate normalization:

$$\bar{q}(t) = \frac{1}{2}\rho V(t)^2, \quad \hat{q}(t) = \frac{q(t)\bar{c}}{2V(t)}$$

This reduces exactly to the linear model when $V(t) \approx U_e$, while still giving a sensible answer further from trim.

Caveats: the derivatives are only valid near the reference condition they were measured at (far excursions need derivative lookup tables vs. Mach/ α); and $\hat{q} = q\bar{c}/(2V)$ is numerically ill-behaved as $V \rightarrow 0$ – guard with a minimum-airspeed floor in code.

Coefficient Buildup: Longitudinal

Each of C_L , C_D , C_m is its zero-angle-of-attack value plus (derivative $\times$ variable), summed over every state and control that affects it. Here α , $\hat{q}$, $\hat{\alpha}$, δ_e denote **absolute** values (angle of attack, non-dimensional rates, and elevator deflection), not perturbations – the subscript e is reserved for trim/equilibrium values used later, in the linearization sections:

Coefficient	Formula
C_L	$C_{L_0} + C_{L_\alpha} \alpha + C_{L_q} \hat{q} + C_{L_{\dot{\alpha}}} \hat{\alpha} + C_{L_{\delta_e}} \delta_e$
C_D	$C_{D_0} + C_{D_\alpha} \alpha + C_{D_{\delta_e}} \delta_e$
C_m	$C_{m_0} + C_{m_\alpha} \alpha + C_{m_q} \hat{q} + C_{m_{\dot{\alpha}}} \hat{\alpha} + C_{m_{\delta_e}} \delta_e$

This compact buildup omits C_{D_q} and $C_{D_{\dot{\alpha}}}$; the general rotation formulas later retain them when data are available. At zero α , normalized rates, and control deflection, only the constant terms remain. Trim is instead found by solving $C_m = 0$ for α_e, δ_{e_e} .

Coefficient Buildup: Lateral-Directional

The same pattern, driven by sideslip and roll/yaw rate instead:

Coefficient	Formula
C_Y	$C_{Y_\beta}\beta + C_{Y_p}\hat{p} + C_{Y_r}\hat{r} + C_{Y_{\delta_a}}\delta_a + C_{Y_{\delta_r}}\delta_r$
C_l	$C_{l_\beta}\beta + C_{l_p}\hat{p} + C_{l_r}\hat{r} + C_{l_{\delta_a}}\delta_a$
C_n	$C_{n_\beta}\beta + C_{n_p}\hat{p} + C_{n_r}\hat{r} + C_{n_{\delta_r}}\delta_r$

$C_{l_{\delta_r}}$ (rudder-into-roll) and $C_{n_{\delta_a}}$ (aileron-into-yaw) are cross-coupling terms, typically small, and are dropped here. At trim ($\beta = 0$, wings level, no control deflection) all three vanish – consistent with $C_{y_e} = C_{l_e} = C_{n_e} = 0$ used implicitly throughout this document.

Which Axes? Where These Coefficients Come From

- ▶ C_L, C_D are defined in the **wind axes** $\mathcal{F}_W$ ($\vec{x}_W$ along the airspeed) – see lect01, “Where Lift and Drag are Defined.”
- ▶ C_Y, C_l, C_m, C_n are defined in the **body-fixed frame** $\mathcal{F}_B$ about the centre of mass — this is the course convention throughout, and the drone data sheet supplies them that way, so no axis conversion applies to them.
- ▶ So only C_L, C_D need rotating: before they can enter the body-axis equations of motion (X, Y, Z, L, M, N used throughout the rest of this course), they must be rotated into $\mathcal{F}_B$ by α – worked out next.

2. Stability Derivatives for Linearized Models

Computing the linearized equations of motion for a given aircraft is a three-step pipeline:

1. **Basic aerodynamic coefficients**

(C_{L_α} , C_{D_α} , C_{m_α} , C_{L_q} , C_{Y_β} , C_{n_β} , ...) from wing/tail/fin geometry.

2. **Nondimensional stability derivatives**

(C_{x_u} , C_{x_α} , C_{z_α} , C_{m_q} , C_{y_β} , C_{l_β} , C_{n_r} , ...) from the basic coefficients, via the wind-to-body rotation of C_L , C_D .

3. **Dimensional derivatives** (X_u , Z_w , M_q , Y_v , L_p , N_r , ...), the entries of the A , B matrices, from the non-dimensional derivatives, via ρ , U_e , S , $\bar{c}$, b .

This section develops the axis conversion, nondimensional derivatives, and dimensional derivatives used in the linearized model.

Wind Axes vs. Body Axes

At trim ($\beta_e = 0$), $\mathbf{C}_{BW}|_{\beta=0} = \mathbf{C}_2(\alpha)$ (lect01, "Where Lift and Drag are Defined: Wind Frame $\mathcal{F}_W$ "). Lift and drag are wind-axis quantities, $\mathbf{F}_W = [-D, Y, -L]^T$. Rotating into body axes via

$\mathbf{F}_B = \mathbf{C}_2(\alpha)\mathbf{F}_W$ gives

$$C_X = C_L \sin \alpha - C_D \cos \alpha, \quad C_Z = -C_L \cos \alpha - C_D \sin \alpha$$

The moment coefficients need no such step: C_l, C_m, C_n are already defined in $\mathcal{F}_B$.

Differentiating C_X, C_Z w.r.t. $u, \alpha, q, \dot{\alpha}, \delta$ at $\alpha = \alpha_e$ gives the nondimensional stability derivatives below.

NASA Aerodynamic-Coefficient Convention

Throughout this reference, the force and moment coefficients describe **aerodynamic loads only**; thus X, Y, Z, L, M, N below exclude propulsion:

$$C_X = \frac{X}{QS}, \quad C_Y = \frac{Y}{QS}, \quad C_Z = \frac{Z}{QS},$$
$$C_l = \frac{L}{QSb}, \quad C_m = \frac{M}{QS\bar{c}}, \quad C_n = \frac{N}{QSb}.$$

Propulsion is modeled separately:

$$C_T = \frac{T}{QS}, \quad C_X^{\text{total}} = C_X + C_T$$

when thrust acts along $+\underline{x}_B$. Gravity is also kept separate in the equations of motion. Thus, C_{x_e} below never includes thrust.

Trim Force Coefficients: General Case

Under the NASA convention, define the aerodynamic, thrust, and weight coefficients separately:

$$Q_e = \frac{1}{2} \rho U_e^2, \quad C_{T_e} = \frac{T_e}{Q_e S}, \quad C_{W_e} = \frac{mg}{Q_e S}.$$

For thrust along $+x_B$ through the center of mass, steady longitudinal trim requires

$$C_{x_e} + C_{T_e} - C_{W_e} \sin \theta_e = 0, \quad C_{z_e} + C_{W_e} \cos \theta_e = 0, \quad C_{m_e} = 0.$$

The aerodynamic values themselves come from the wind-to-body-axis rotation:

$$C_{x_e} = C_{L_e} \sin \alpha_e - C_{D_e} \cos \alpha_e,$$

$$C_{z_e} = -C_{L_e} \cos \alpha_e - C_{D_e} \sin \alpha_e.$$

Here $\theta_e = \gamma_e + \alpha_e$; do not set $\theta_e = 0$ unless the chosen trim condition actually has zero pitch attitude.

Longitudinal Stability-Derivative Definitions (1/2)

Define the nondimensional perturbations

$$\hat{u} = \frac{u}{U_e}, \quad \hat{q} = \frac{q\bar{c}}{2U_e}, \quad \hat{\alpha} = \frac{\dot{\alpha}\bar{c}}{2U_e}.$$

Each stability derivative is a partial derivative evaluated at equilibrium, with the other independent variables held fixed:

Perturbation	Axial force	Normal force	Pitching moment
$\hat{u}$	$C_{x_u} \equiv \left. \frac{\partial C_x}{\partial \hat{u}} \right _e$	$C_{z_u} \equiv \left. \frac{\partial C_z}{\partial \hat{u}} \right _e$	$C_{m_u} \equiv \left. \frac{\partial C_m}{\partial \hat{u}} \right _e$
α	$C_{x_\alpha} \equiv \left. \frac{\partial C_x}{\partial \alpha} \right _e$	$C_{z_\alpha} \equiv \left. \frac{\partial C_z}{\partial \alpha} \right _e$	$C_{m_\alpha} \equiv \left. \frac{\partial C_m}{\partial \alpha} \right _e$
$\hat{q}$	$C_{x_q} \equiv \left. \frac{\partial C_x}{\partial \hat{q}} \right _e$	$C_{z_q} \equiv \left. \frac{\partial C_z}{\partial \hat{q}} \right _e$	$C_{m_q} \equiv \left. \frac{\partial C_m}{\partial \hat{q}} \right _e$

By convention, the subscripts omit the hats: for example, C_{z_q} means $\partial C_z / \partial \hat{q}$, not $\partial C_z / \partial q$.

Longitudinal Stability-Derivative Definitions (2/2)

The remaining longitudinal derivatives are

Perturbation	Axial force	Normal force	Pitching moment
$\hat{\alpha}$	$C_{x_{\hat{\alpha}}} \equiv \left. \frac{\partial C_x}{\partial \hat{\alpha}} \right _e$	$C_{z_{\hat{\alpha}}} \equiv \left. \frac{\partial C_z}{\partial \hat{\alpha}} \right _e$	$C_{m_{\hat{\alpha}}} \equiv \left. \frac{\partial C_m}{\partial \hat{\alpha}} \right _e$
δ_e	$C_{x_{\delta_e}} \equiv \left. \frac{\partial C_x}{\partial \delta_e} \right _e$	$C_{z_{\delta_e}} \equiv \left. \frac{\partial C_z}{\partial \delta_e} \right _e$	$C_{m_{\delta_e}} \equiv \left. \frac{\partial C_m}{\partial \delta_e} \right _e$

Thus, to first order, for $i \in \{x, z, m\}$,

$$\Delta C_i \approx C_{i_u} \hat{u} + C_{i_\alpha} \Delta \alpha + C_{i_q} \hat{q} + C_{i_{\hat{\alpha}}} \hat{\alpha} + C_{i_{\delta_e}} \Delta \delta_e.$$

Since $\Delta \alpha \simeq w/U_e$, one may equivalently write

$C_{i_w} \equiv \partial C_i / \partial (w/U_e)|_e \simeq C_{i_\alpha}$. All C_i above are aerodynamic only; thrust derivatives such as C_{T_u} remain separate under the NASA convention.

Longitudinal: General Formulas (1/2)

Differentiate the exact body-axis rotation

$$C_X = C_L \sin \alpha - C_D \cos \alpha, \quad C_Z = -C_L \cos \alpha - C_D \sin \alpha$$

at the actual trim angle α_e . For angle of attack,

$$C_{X_\alpha} = C_{L_\alpha} \sin \alpha_e + C_{L_e} \cos \alpha_e - C_{D_\alpha} \cos \alpha_e + C_{D_e} \sin \alpha_e$$

$$C_{Z_\alpha} = -C_{D_\alpha} \sin \alpha_e - C_{D_e} \cos \alpha_e - C_{L_\alpha} \cos \alpha_e + C_{L_e} \sin \alpha_e$$

$$C_{m_\alpha} = \left. \frac{\partial C_m}{\partial \alpha} \right|_e.$$

These expressions retain every $\sin \alpha_e$ and $\cos \alpha_e$ term and therefore apply at arbitrary trim angle of attack within the linearization.

Longitudinal: General Formulas (2/2)

For $\xi \in \{u, q, \dot{\alpha}, \delta_e\}$, where rate derivatives are taken with respect to their normalized variables, the exact rotation gives

$$\begin{aligned}C_{x_\xi} &= C_{L_\xi} \sin \alpha_e - C_{D_\xi} \cos \alpha_e, \\C_{z_\xi} &= -C_{L_\xi} \cos \alpha_e - C_{D_\xi} \sin \alpha_e.\end{aligned}$$

The pitching-moment coefficient is unchanged by this rotation, so each C_{m_ξ} carries over directly. Thus the formulas retain possible drag and lift dependence on speed, pitch rate, angle-of-attack rate, and elevator deflection. No terms are removed by assuming small α_e , low Mach number, or negligible drag coupling.

Lateral Stability-Derivative Definitions (1/2)

Define the nondimensional lateral-directional perturbations

$$\beta \simeq \frac{v}{U_e}, \quad \hat{p} = \frac{pb}{2U_e}, \quad \hat{r} = \frac{rb}{2U_e}.$$

As in the longitudinal case, every derivative is evaluated at equilibrium with the other independent variables held fixed:

Perturbation	Side force	Rolling moment	Yawing moment
β	$C_{y_\beta} \equiv \left. \frac{\partial C_y}{\partial \beta} \right _e$	$C_{l_\beta} \equiv \left. \frac{\partial C_l}{\partial \beta} \right _e$	$C_{n_\beta} \equiv \left. \frac{\partial C_n}{\partial \beta} \right _e$
$\hat{p}$	$C_{y_p} \equiv \left. \frac{\partial C_y}{\partial \hat{p}} \right _e$	$C_{l_p} \equiv \left. \frac{\partial C_l}{\partial \hat{p}} \right _e$	$C_{n_p} \equiv \left. \frac{\partial C_n}{\partial \hat{p}} \right _e$
$\hat{r}$	$C_{y_r} \equiv \left. \frac{\partial C_y}{\partial \hat{r}} \right _e$	$C_{l_r} \equiv \left. \frac{\partial C_l}{\partial \hat{r}} \right _e$	$C_{n_r} \equiv \left. \frac{\partial C_n}{\partial \hat{r}} \right _e$

The conventional subscripts again omit the hats: C_{l_p} denotes $\partial C_l / \partial \hat{p}$, not $\partial C_l / \partial p$.

Lateral Stability-Derivative Definitions (2/2)

Control derivatives are defined with respect to the control-surface deflections:

Perturbation	Side force	Rolling moment	Yawing moment
δ_a	$C_{y\delta_a} \equiv \left. \frac{\partial C_y}{\partial \delta_a} \right _e$	$C_{l\delta_a} \equiv \left. \frac{\partial C_l}{\partial \delta_a} \right _e$	$C_{n\delta_a} \equiv \left. \frac{\partial C_n}{\partial \delta_a} \right _e$
δ_r	$C_{y\delta_r} \equiv \left. \frac{\partial C_y}{\partial \delta_r} \right _e$	$C_{l\delta_r} \equiv \left. \frac{\partial C_l}{\partial \delta_r} \right _e$	$C_{n\delta_r} \equiv \left. \frac{\partial C_n}{\partial \delta_r} \right _e$

Thus, to first order, for $i \in \{y, l, n\}$,

$$\Delta C_i \approx C_{i\beta} \Delta \beta + C_{i_p} \hat{p} + C_{i_r} \hat{r} + C_{i\delta_a} \Delta \delta_a + C_{i\delta_r} \Delta \delta_r.$$

Since $\Delta \beta \simeq v/U_e$, one may equivalently write

$C_{i_v} \equiv \partial C_i / \partial (v/U_e)|_e \simeq C_{i\beta}$. The table includes all cross-coupling derivatives; simplified models often neglect small terms such as $C_{l\delta_r}$ and $C_{n\delta_a}$.

Lateral-Directional: Which Axes?

Unlike C_L , C_D , the lateral coefficients need no axis conversion in this course: C_Y , C_l , C_n and every derivative built from them are defined directly in $\mathcal{F}_B$, about the centre of mass, and the drone data sheet supplies them that way.

- ▶ $C_{y\beta}$, $C_{l\beta}$, $C_{n\beta}$ use $\beta \simeq v/U_e$ with v the body-axis side velocity;
- ▶ C_{yp} , C_{lp} , C_{np} and C_{yr} , C_{lr} , C_{nr} use $\hat{p} = pb/2U_e$ and $\hat{r} = rb/2U_e$ with p, r the body-axis rates;
- ▶ the control derivatives use the physical deflections δ_a, δ_r .

If you ever import derivatives from a source that tabulates them in *stability axes* (a body-fixed frame rotated from $\mathcal{F}_B$ by the trim α_e — Etkin's appendices and most wind-tunnel data are like this), they must be rotated into $\mathcal{F}_B$ first, and the inertias I_{xx} , I_{zz} , I_{xz} rotate with them. That conversion is outside the scope of this course; the data used here is body-axis throughout.

From Nondimensional to Dimensional: Longitudinal

The axial column gives the derivative of the total axial force $X_{\text{total}} = X_a + T$; the Z and M columns remain aerodynamic for thrust along $\vec{x}_B$ through the center of mass. Assume

$$C_T = C_T(\hat{u}, \delta_p).$$

	X_{total} (N)	Z (N)	M (N·m)
u	$\frac{1}{2}\rho U_e S [C_{x_u} + C_{T_u} + 2(C_{x_e} + C_{T_e})]$	$\frac{1}{2}\rho U_e S (C_{z_u} + 2C_{z_e})$	$\frac{1}{2}\rho U_e S \bar{c} (C_{m_u} + 2C_{m_e})$
α	$\frac{1}{2}\rho U_e^2 S C_{x_\alpha}$	$\frac{1}{2}\rho U_e^2 S C_{z_\alpha}$	$\frac{1}{2}\rho U_e^2 S \bar{c} C_{m_\alpha}$
w	$\frac{1}{2}\rho U_e S C_{x_\alpha}$	$\frac{1}{2}\rho U_e S C_{z_\alpha}$	$\frac{1}{2}\rho U_e S \bar{c} C_{m_\alpha}$
q	$\frac{1}{4}\rho U_e S \bar{c} C_{x_q}$	$\frac{1}{4}\rho U_e S \bar{c} C_{z_q}$	$\frac{1}{4}\rho U_e S \bar{c}^2 C_{m_q}$
$\dot{\alpha}$	$\frac{1}{4}\rho U_e S \bar{c} C_{x_{\dot{\alpha}}}$	$\frac{1}{4}\rho U_e S \bar{c} C_{z_{\dot{\alpha}}}$	$\frac{1}{4}\rho U_e S \bar{c}^2 C_{m_{\dot{\alpha}}}$
δ_e	$\frac{1}{2}\rho U_e^2 S C_{x_{\delta_e}}$	$\frac{1}{2}\rho U_e^2 S C_{z_{\delta_e}}$	$\frac{1}{2}\rho U_e^2 S \bar{c} C_{m_{\delta_e}}$

C_{x_u}, C_{x_e} remain aerodynamic-only quantities; C_{T_u}, C_{T_e} are added explicitly to obtain the final X_u^{total} . $M_q, M_{\dot{\alpha}}$ carry an extra factor of $\bar{c}$ (vs. X, Z) from the moment arm. The w -row is the α -row divided by U_e , from $\alpha \approx w/U_e$ – use whichever row matches the chosen state vector.

Propulsive Derivatives Remain Separate

For the quasi-steady engine model $T = T(U, \delta_p)$, linearization at trim gives

$$\Delta T = T_u u + T_{\delta_p} \Delta \delta_p, \quad T_u = \left. \frac{\partial T}{\partial U} \right|_e, \quad T_{\delta_p} = \left. \frac{\partial T}{\partial \delta_p} \right|_e.$$

If thrust is instead tabulated as $C_T(\hat{u}, \delta_p)$, then

$$T_u = \frac{Q_e S}{U_e} (C_{T_u} + 2C_{T_e}), \quad T_{\delta_p} = Q_e S C_{T_{\delta_p}}.$$

With thrust along $+\vec{x}_B$ through the center of mass,

$$\Delta X_{\text{total}} = \Delta X_a + \Delta T,$$

while the aerodynamic derivatives $X_u, X_w, \dots$ remain unchanged.

From Nondimensional to Dimensional: Lateral-Directional

Same pattern, with span b in place of $\bar{c}$ (since $\hat{p}, \hat{r}$ are nondimensionalized by $b/2U_e$), and the state kept dimensional ($v = U_e\beta$, p , r – not $\beta, \hat{p}, \hat{r}$):

	Y (N)	L (N·m)	N (N·m)
v	$\frac{1}{2}\rho U_e S C_{y\beta}$	$\frac{1}{2}\rho U_e S b C_{l\beta}$	$\frac{1}{2}\rho U_e S b C_{n\beta}$
p	$\frac{1}{4}\rho U_e S b C_{yp}$	$\frac{1}{4}\rho U_e S b^2 C_{lp}$	$\frac{1}{4}\rho U_e S b^2 C_{np}$
r	$\frac{1}{4}\rho U_e S b C_{yr}$	$\frac{1}{4}\rho U_e S b^2 C_{lr}$	$\frac{1}{4}\rho U_e S b^2 C_{nr}$
δ_a	$\frac{1}{2}\rho U_e^2 S C_{y\delta_a}$	$\frac{1}{2}\rho U_e^2 S b C_{l\delta_a}$	$\frac{1}{2}\rho U_e^2 S b C_{n\delta_a}$
δ_r	$\frac{1}{2}\rho U_e^2 S C_{y\delta_r}$	$\frac{1}{2}\rho U_e^2 S b C_{l\delta_r}$	$\frac{1}{2}\rho U_e^2 S b C_{n\delta_r}$

All lateral coefficients here are body-axis quantities, so no axis conversion enters this table — only the ρ , U_e , S , b scaling.