

Kinematics

Closed-Form Body-Rate Transformation

Acknowledgement: These slides were developed with reference to course materials by Professor Hugh H. T. Liu.

Contents

1 Kinematics

1. Kinematics

- ▶ The closed-form inverse of $\mathbf{S}_B$.

The closed-form inverse of $\mathbf{S}_B$

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin \theta \\ 0 & \cos \phi & \sin \phi \cos \theta \\ 0 & -\sin \phi & \cos \phi \cos \theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \triangleq \mathbf{S}_B \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

Find the closed-form inverse of the coordinate transformation matrix:

$$\begin{bmatrix} \underline{x}_B & \underline{y}_2 & \underline{z}_1 \end{bmatrix} = \begin{bmatrix} \underline{x}_B & \underline{y}_B & \underline{z}_B \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & -\sin \theta \\ 0 & \cos \phi & \cos \theta \sin \phi \\ 0 & -\sin \phi & \cos \theta \cos \phi \end{bmatrix}}_{\text{coordinate transformation matrix}} \quad (1)$$

$$\underline{x}_2 = \underline{x}_B \quad (2)$$

$$\underline{y}_2 = \cos \phi \underline{y}_B - \sin \phi \underline{z}_B \quad (3)$$

$$\underline{z}_1 = -\sin \theta \underline{x}_B + \cos \theta \sin \phi \underline{y}_B + \cos \theta \cos \phi \underline{z}_B \quad (4)$$

Multiplying both sides of (3) by $\cos \theta \sin \phi$ and both sides of (4) by $\cos \phi$ gives

$$\cos \theta \sin \phi \underline{y}_2 = \cos \theta \sin \phi \cos \phi \underline{y}_B - \cos \theta \sin^2 \phi \underline{z}_B$$

$$\cos \phi \underline{z}_1 = -\cos \phi \sin \theta \underline{x}_B + \cos \phi \cos \theta \sin \phi \underline{y}_B + \cos \theta \cos^2 \phi \underline{z}_B \quad (5)$$

Subtracting the first of these from the second,

$$\begin{aligned}\cos \phi \underline{z}_1 - \cos \theta \sin \phi \underline{y}_2 &= -\cos \phi \sin \theta \underline{x}_B + \cos \theta \underline{z}_B \\ \rightarrow \underline{z}_B &= \cos \phi \tan \theta \underline{x}_B - \sin \phi \underline{y}_2 + \cos \phi \sec \theta \underline{z}_1\end{aligned}\quad (6)$$

For $\underline{y}_B$, we divide both sides of (3) by $\cos \phi$, and obtain the following:

$$\underline{y}_B = \frac{\underline{y}_2}{\cos \phi} + \tan \phi \underline{z}_B \quad (7)$$

Plugging in the expression of $\underline{z}_B$ from (6), we have

$$\begin{aligned}\underline{y}_B &= \frac{\underline{y}_2}{\cos \phi} + \tan \phi (\cos \phi \tan \theta \underline{x}_B - \sin \phi \underline{y}_2 + \cos \phi \sec \theta \underline{z}_1) \\ &= \sin \phi \tan \theta \underline{x}_B + \cos \phi \underline{y}_2 + \sin \phi \sec \theta \underline{z}_1\end{aligned}\quad (8)$$

$$\begin{bmatrix} \underline{x}_B & \underline{y}_B & \underline{z}_B \end{bmatrix} = \begin{bmatrix} \underline{x}_B & \underline{y}_2 & \underline{z}_1 \end{bmatrix} \begin{bmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi \sec \theta & \cos \phi \sec \theta \end{bmatrix} \quad (9)$$

$$\underline{\omega} = \dot{\psi} \underline{z}_1 + \dot{\theta} \underline{y}_2 + \dot{\phi} \underline{x}_B$$

$$\begin{bmatrix} \underline{x}_B & \underline{y}_B & \underline{z}_B \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \underline{x}_B & \underline{y}_2 & \underline{z}_1 \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \underline{x}_B & \underline{y}_2 & \underline{z}_1 \end{bmatrix} \begin{bmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi \sec \theta & \cos \phi \sec \theta \end{bmatrix}}_{\mathbf{S}_B^{-1}} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \underline{x}_B & \underline{y}_2 & \underline{z}_1 \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$